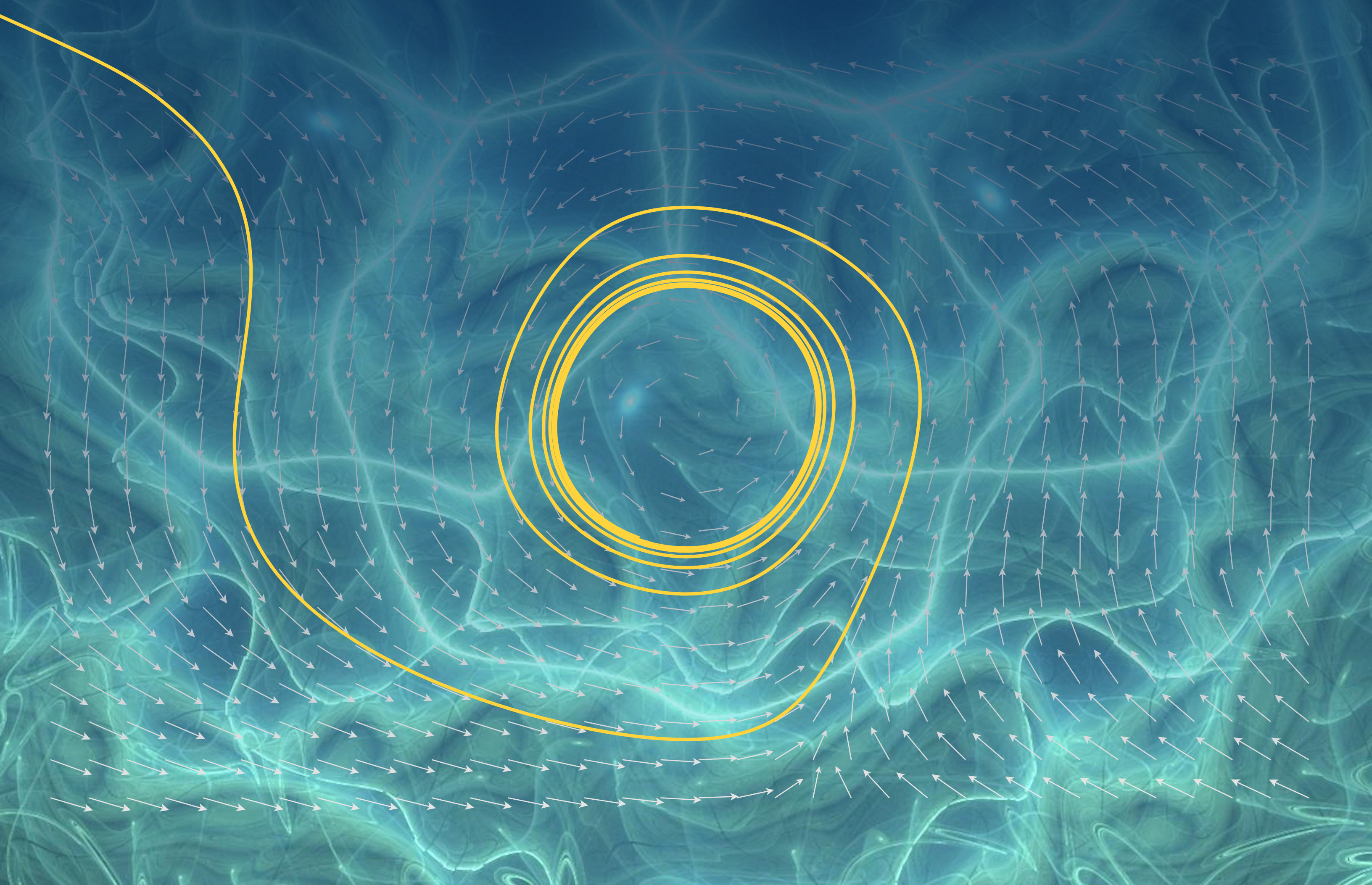
Differential Equations

MAT244 Workbook

2025/06/26 Edition



Jason Siefken
Bernardo Galvão-Sousa

Table of Contents (Modules)

Modelling

In this module you will learn

- What a differential equation is.
- What a solution to a differential equation is.
- The difference between a discrete model and a differential-equation based model.

Modelling

Suppose you are observing some green ants walking on the sidewalk. In the first minute you record 10 ants. In the second, you record 20. In the third, 40. This continues until there are too many ants for you to count.

Minute	#Green Ants
1	10
2	20
3	40
4	80
⋮	⋮

Since you lost count of the ants, you decide to use mathematics to predict how many ants walked by on minutes 5, 6, ... You notice the pattern that

$$\text{Green ants per minute } n = 2^{n-1} \cdot 10.$$

Stupendous! Mathematics now predicts there were 160 ants during minute 5. But something else catches your eye. Across the sidewalk are brown ants. You count these ants every minute.

Minute	#Brown Ants
1	3
2	6
3	12
4	24
⋮	⋮

The pattern is slightly different. This time,

$$\text{Brown ants per minute } n = 2^{n-1} \cdot 3.$$

Your friend, who was watching you the whole time, looks confused. They question: "Why come up with two complicated equations when you can describe both types of ant at once?"

$$\#\text{Ants at minute } n = 2 \cdot (\#\text{Ants at minute } n-1)$$
$$\#\text{Green ants at minute } 1 = 10$$
$$\#\text{Brown ants at minute } 1 = 3$$

Your friend has a point. Their model is elegant, but your model can predict how many ants pass by at minute 3.222! (Though, your friend would probably complain that 46.654 is not a number of ants....) You and your friend have just come up with two different mathematical models for the number of ants that walk across the sidewalk. They happen to make similar predictions for each minute and each have their strengths and weaknesses. In this text, we will be focused on third type of mathematical model—one that uses differential equations at its core.

Types of Models

Mathematical Model. A *mathematical model* is a description of the world that is

1. created in the service of answering a question, and
2. where the complexity of the world has been abstracted away to numbers, quantities, and their relationships.[1]

In the previous situation, the *question* you were trying to answer was "how many ants are there at a given minute?". We sidestepped difficult issues like, "Is an ant that is missing three legs still an ant?" by using the common-sense conventions and so we could use single numbers to represent our quantity of interest (the ants).

You and your friend already came up with two types of models.

- An **explicit** model based on known functions.
- A **recursive** model where subsequent terms are based on previous terms and initial conditions.

If we define $A(n)$ to be the number of ants crossing the sidewalk at minute n, the *explicit* model presented for green ants is $A(n) = 2^{n-1} \cdot 10$ and the *recursive* model presented is $A(n) = 2 \cdot A(n-1)$ with $A(1) = 10$.

Each model has pros and cons. For example, the explicit model allows you to calculate the number of ants at any minute with few button presses on a calculator, whereas the recursive model is more difficult to calculate but makes it clear that the number of ants is doubling every minute.

Often times recursive models are easier to write down than explicit models[2], but they may be harder to analyze. A third type of model has similarities to both explicit and recursive models but adds more by bringing the power of calculus to modeling.

- A **differential-equations** model is a model based on a relationship between a function's derivative(s), its values, and an initial condition.

Differential-equations models are useful because derivatives correspond to rates of change, and things in the world are always changing. Let's try to come up with a differential equations model for *brown* ants.

We'd like an equation relating $A(n)$, the number of brown ants at minute n, to $A'(n)$, the *instantaneous rate of change* of the number of ants at minute n. Making a table, we see

Minute	#Brown Ants	Change (from prev. minute)
1	3	?
2	6	3
3	12	6
4	24	12

or

[1] Other mathematical objects are also allowed.

[2] In fact, in many real-world situations, an explicit model in terms of already known functions doesn't exist.

Minute	#Brown Ants	Change (from next minute)
1	3	3
2	6	6
3	12	12
4	24	?

depending on whether we record the change from the previous minute or up to the subsequent minute. Neither table gives the *instantaneous* rate of change, but in both tables, the change is proportional to the number of ants. So, we can set up a model

$$A'(n) = kA(n)$$

where k is a constant of proportionality that we will try to determine later. We've just written down a *differential equation* with an undetermined parameter k.

Differential Equation. A *differential equation* is an equation relating a function to one or more of its derivatives.

We'd like to figure out what k is. One way is to solve the differential equation (that is, find an explicit function which satisfies the differential equation) and find which values of k make our model correctly predict the data. This is called *fitting* the model to data.

Note that in general, fitting a model to data doesn't necessarily produce *unique* values for the unknown parameters, and a fitted model (especially when the data comes from real-world observations) usually doesn't reproduce the data exactly. However, in the case of these ants, we just might get lucky.

Solving Differential Equations

Our approach to fitting will be to first *solve* the differential equation, i.e. find what functions A satisfy the equation $A'(t) = kA(t)$, and then see which values of k produce solutions that fit the data. Our primary method for finding explicit solutions to differential equations will be guess-and-check.

Example. Use guess-and-check to solve $A'(n) = kA(n)$.

Since $A' \approx A$, we might start with a function that is equal to its own derivative. There is a famous one: e^n. Testing, we see $\frac{\mathrm{d}}{\mathrm{d}n}e^n = e^n = ke^n$ if $k = 1$, but it doesn't work for other k's. Trying e^{kn} instead yields $\frac{\mathrm{d}}{\mathrm{d}n}e^{kn} = ke^{kn}$ which holds for all k. Thus $A(n) = e^{kn}$ is *a* solution to $A'(n) = kA(n)$. However, there are other solutions. Because $\frac{\mathrm{d}}{\mathrm{d}n}Ce^{kn} = C\big(ke^{kn}\big) = k\big(Ce^{kn}\big)$ for every number C, the function $A(n) = Ce^{kn}$ is a solution to $A'(n) = kA(n)$.

By guessing-and-checking, we have found an infinite number of solutions to $A'(n) = kA(n)$.

It's now time to fit our model to the data.

Example. Find values of C and k so that $A(n) = Ce^{kn}$ best models brown ants.

Taking two rows from our brown ants table, we see

$$\begin{aligned} A(1) &= Ce^{k} &= 3 \\ A(2) &= Ce^{2k} &= 6 \end{aligned}$$

Since e^k can never be zero, from the first equation we get $C = \frac{3}{e^k}$. Combining with the second equation we find

$$Ce^{2k} = \frac{3}{e^k}e^{2k} = 3e^k = 6$$

and so $e^k = 2$. In other words $k = \ln 2$. Plugging this back in, we find $C = \frac{3}{2}$. Thus our fitted model is

$$A(n) = \tfrac{3}{2}e^{n\ln 2}.$$

Upon inspection, we can see that $\frac{3}{2}e^{n\ln 2} = 3 \cdot 2^{n-1}$, which is the explicit model that was first guessed for brown ants.

General Solutions

Solution. A ***solution*** to a differential equation on a domain D is a function with domain D which satisfies the differential equation on D. (The domain D is usually take to be a connected set like an interval.)

A given differential equation can have many solutions.

The *solution set*, *complete solution*, or *general solution* is the family of all functions that satisfy the differential equation on a given domain.

For example, we can easily verify that the functions $y(t) = e^{2t}$ and $y(t) = 17e^{2t}$ both satisfy the differential equation $y' = 2y$. Any multiple of one of these solutions is also a solution, so the set

$$\{f : f(t) = Ce^{2t} \text{ for some } C \in \mathbb{R}\}$$

is a set of solutions to the differential equation $y' = 2y$. Because there are no other solutions, $\{f : f(t) = Ce^{2t} \text{ for some } C \in \mathbb{R}\}$ is, in fact, the *solution set/complete solution* to the differential equation.

Writing General Solutions

Because set notation can be cumbersome, we often write solutions in an abbreviated form. For example, we might write the general solution to $y' = 2y$ as

$$y(t) = Ce^{2t} \qquad \text{where } C \in \mathbb{R} \text{ is a parameter.}$$

Important note: We always specify which terms/variables in a general solution are parameters (e.g., by writing "where ... is a parameter"). This is because differential equations coming from models often involve many variables. It's important to distinguish which variables come from modelling assumptions and which are free parameters of your solution.

For example, when finding the general solution to $y' = ky$, if we wrote "$y(t) = Ce^{kt}$", it wouldn't be clear if every choice of C gives a valid solution, if every choice of k gives a valid solution, or both. Here, k comes from the equation/model and we are not allowed to choose it when solving. However, C can be chosen by us and every choice results in a valid solution.

Initial Value Problems

Out of the infinitely many solutions to a differential equation, we usually only want a few of them. One way to winnow down the solution set is to specify an *initial condition*: a value that the solution to the differential equation attains at some point/time.

For example, suppose $P' = 2P$ models a population and we know the population starts at 100 at time $t = 0$ (i.e., $P(0) = 100$). The general solution to $P' = 2P$ is $P(t) = Ce^{2t}$ where C is a parameter, but only one solution in this family satisfies $P(0) = 100$. (Stop now and figure out what value of C leads to a solution satisfying $P(0) = 100$.)

Solving Methods

There are a number of algorithms and techniques to find explicit solutions to particular classes of differential equations. These include:

- *Separation of variables*, a technique applicable to differential equations of the form $x'(t) = F(x(t)) \cdot G(t)$. (See Appendix B)
- *Integrating factors*, a technique applicable to differential equations of the form $x'(t) + f(t) \cdot x(t) = g(t)$. (See Appendix C)
- *Series solutions*, a technique to express the solution to a differential equation as a power series (e.g., $f(t) = \sum_{n=0}^{\infty} a_n t^n$). (See Appendix D)

Unfortunately, despite these techniques, **there is no algorithm for explicitly solving a general differential equation**. But, it is easy to check whether a particular function is a solution to a differential equation, since there *is* an algorithm to differentiate functions.[3] This means **guess and check** is an effective method for finding explicit solutions to differential equations, and, though we will learn some other solving techniques, guess and check will be our go-to method.

Practice Problems

1. Explain what you need to do in two different ways.
   (a) Way 1
   (b) Way 2
2. A second question.
3. A third question.

Solutions for Module 1

1. (a) Way 1 is good
   (b) Way 2 is better
2. The answer to the second question.
3.

[3]More specifically, there is an algorithm to differentiate the *elementary* functions, those functions formed by compositions, sums, products, and quotients of polynomials, trig, exponentials, and logs.

1 You are observing starfish that made their way to a previously uninhabited tide-pool. You'd like to predict the year-on-year population of these starfish. You start with a simple assumption

$$\#\text{new children per year} \sim \text{size of current population}$$

1.1 Come up with a mathematical model for the number of star fish in a given year. Your model should

- Define any notation (variables and parameters) you use
- Include at least one formula/equation
- Explain how your formula/equation relates to the starting assumption

2 Let

(Birth Rate) $K = 1.1$ children per starfish per year
(Initial Pop.) $P_0 = 10$ star fish

and define the model $\mathbf{M}_1$ to be the model for starfish population with these parameters.

2.1 Simulate the total number of starfish per year using Excel.

3 Recall the model $\mathbf{M}_1$ (from the previous question).

Define the model $\mathbf{M}_1^*$ to be

$$P(t) = P_0 e^{0.742t}$$

3.1 Are $\mathbf{M}_1$ and $\mathbf{M}_1^*$ different models or the same?

3.2 Which of $\mathbf{M}_1$ or $\mathbf{M}_1^*$ is better?

3.3 List an advantage and a disadvantage for each of $\mathbf{M}_1$ and $\mathbf{M}_1^*$.

4 In the model $\mathbf{M}_1$, we assumed the starfish had K children at one point during the year.

4.1 Create a model $\mathbf{M}_n$ where the starfish are assumed to have K/n children n times per year (at regular intervals).

4.2 Simulate the models $\mathbf{M}_1$, $\mathbf{M}_2$, $\mathbf{M}_3$ in Excel. Which grows fastest?

4.3 What happens to $\mathbf{M}_n$ as $n \to \infty$?

5 **Exploring $\mathbf{M}_n$**

We can rewrite the assumptions of $\mathbf{M}_n$ as follows:

- At time t there are $P_n(t)$ starfish.
- $P_n(0) = 10$
- During the time interval $\left(t, t + \frac{1}{n}\right)$ there will be (on average) $\frac{K}{n}$ new children per starfish.

5.1 Write an expression for $P_n\left(t + \frac{1}{n}\right)$ in terms of $P_n(t)$.

5.2 Write an expression for ΔP_n, the change in population from time t to $t + \Delta t$.

5.3 Write an expression for $\frac{\Delta P_n}{\Delta t}$.

5.4 Write down a differential equation relating $P'(t)$ to $P(t)$ where $P(t) = \lim_{n\to\infty} P_{n(t)}$.

6

Recall the model $\mathbf{M}_1$ defined by:

- $P_1(0) = 10$
- $P_1(t+1) = KP(t)$ for $t \geq 0$ years and $K = 1.1$.

Define the model $\mathbf{M}_\infty$ by:

- $P(0) = 10$
- $P'(t) = kP(t)$.

6.1 If $k = K = 1.1$, does the model $\mathbf{M}_\infty$ produce the same population estimates as $\mathbf{M}_1$?

7 Suppose that the estimates produced by $\mathbf{M}_1$ agree with the actual (measured) population of starfish.

Fill out the table indicating which models have which properties.

Model	Accuracy	Explanatory	(your favourite property)
$\mathbf{M}_1$			
$\mathbf{M}_1^*$			
$\mathbf{M}_\infty$			

Recall the model $\mathbf{M}_1$ defined by:

- $P_1(0) = 10$
- $P_1(t+1) = KP(t)$ for $t \geq 0$ years and $K = 1.1$.

Define the model $\mathbf{M}_\infty$ by:

- $P(0) = 10$
- $P'(t) = kP(t)$.

8.1 Suppose that $\mathbf{M}_1$ accurately predicts the population. Can you find a value of k so that $\mathbf{M}_\infty$ accurately predicts the population?

9 After more observations, scientists notice a seasonal effect on starfish. They propose a new model called **S**:

- $P(0) = 10$
- $P'(t) = k \cdot P(t) \cdot |\sin(2\pi t)|$

9.1 What can you tell about the population (without trying to compute it)?

9.2 Assuming $k = 1.1$, estimate the population after 10 years.

9.3 Assuming $k = 1.1$, estimate the population after 10.3 years.

Simulation

In this module you will learn

- How to use slope fields and Euler's method to approximate solutions to differential equations.

Most differential equations do not have explicit *elementary* solutions. That is, solutions which can be written in terms of polynomials, exponentials, logarithms, etc.. Put another way: most differential equations do not have "nice" solutions. However, most ordinary differential equations that you will encounter *will have solutions* (even though you cannot write them down).

There are two approaches when a differential equation cannot be explicitly "solved":

1. Make up a new name for a function *defined* to be the solution.
2. Approximate the solution.

Before the days of computers, approach 1 was commonplace. If you've heard of Bessel functions or Airy functions, these are functions defined to be the solutions to particular differential equations.[4]

In this course, we will focus on approach 2.

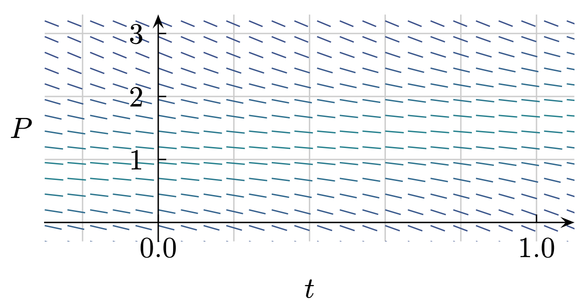
Visual Estimation

Suppose that $P(t)$ models the number of bacteria (in billions) in a petri dish at time t. (Since the bacteria are so numerous, it makes sense to think of $P(t)$ as continuous.) Further, suppose that P satisfies

$$P'(t) = \sin\left(\frac{\pi}{2} \cdot P(t) - t\right) - 1.5 \qquad \text{and} \qquad P(0) = 3.$$

Our task is to approximate $P(1)$.

We may not know exactly what our solution curve is, but the differential equation tells us what tangent lines to the solution curve look like. It stands to reason that if we draw little segments of tangent lines all over the (t, P)–plane, we can visually guess at the shape of the solution curve.



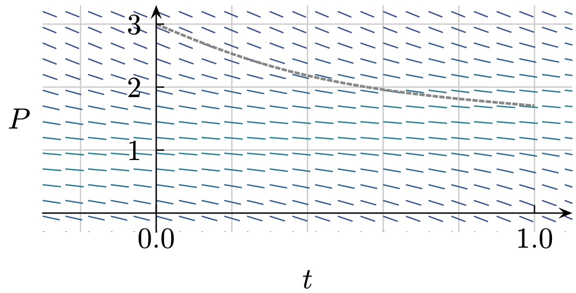
The above diagram, where small segments of tangent lines to solutions populate the plane, is called a *slope field*.

Slope Field. A ***slope field*** for the differential equation $y' = f(x, y)$ is a collection of line segments that are tangent to different solutions of the differential equation.

The line segments are generated by evaluating $f(x, y)$ over a rectangular grid of points, and at each point (x, y) of the grid creating a line segment of slope $f(x, y)$. The length of the line segments is not important, but should be consistent.

Starting at $(t, P(t)) = (0, 3)$, we can trace out what an approximate solution curve looks like.

[4]Even the humble exponential is often defined to be the solution to $f'(t) = f(t)$ which satisfies $f(0) = 1$.



The solution curve we draw should be tangent to every line segment it passes through. In the drawing above, the solution curve is decreasing and slightly concave up. Based on the drawing, we can estimate $P(1) \approx 1.7$.

Tracing out solutions on slope fields is a good way to see overall qualities of a solution, like whether it is increasing/decreasing or its concavity. But, its numerical accuracy is limited. A small change in how we sketched the curve could lead to estimates that $P(1) \approx 1.5$ or $P(1) \approx 1.9$, a wide range. If we're after precise numerical estimates, we can formalize this "slope field estimation" by means of Euler's method.

Euler's Method

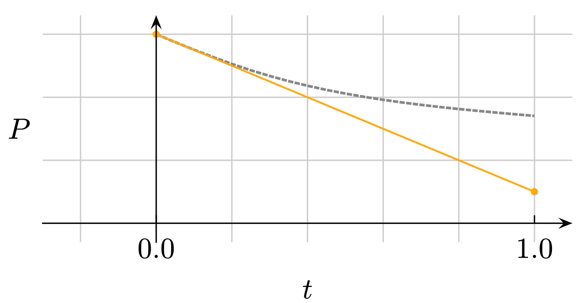
Recall that $P(t)$ models the number of bacteria (in billions) in a petri dish at time t with

$$P'(t) = \sin\left(\frac{\pi}{2} \cdot P(t) - t\right) - 1.5 \qquad \text{and} \qquad P(0) = 3.$$

Our task is to numerically approximate $P(1)$.

A simple approach would be to approximate $P(1)$ using a tangent line through the point $(0, P(0))$. We know the slope $P'(0) = -2.5$, and so we get an approximation of

$$P(1) \approx P(0) + P'(0)(t - 0)\Big|_{t=1} = 0.5.$$



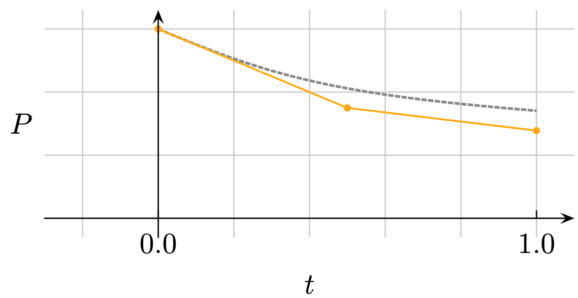
Our approximation is an underestimate. How can we be more accurate? By using two tangent lines!

A tangent line is a good approximation to a function near the point of tangency. So, instead of approximating $P(1)$ using a tangent line at $t = 0$, let's approximate $P(0.5)$ using a tangent line at $t = 0$ and then approximate $P(1)$ using a tangent line at $t = 0.5$.

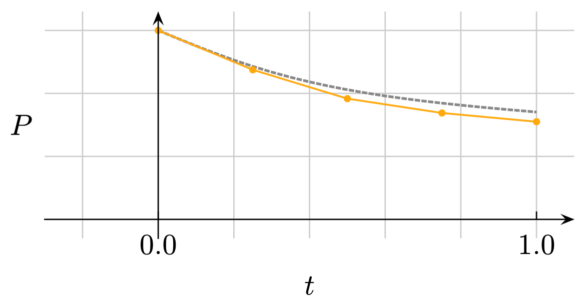
Unfortunately, we won't know the exact value of $P(0.5)$, so we cannot use the *true* tangent line to the solution curve at $t = 0.5$. But, we know an approximate value of $P(0.5)$. For now, we will assume this approximate value is close enough.

$$P_{\text{approx}}(0.5) = P(0) \qquad + P'(0)\cdot(t-0)\Big|_{t=\frac{1}{2}} \qquad = \underbrace{1.75}_{\substack{\text{Estimate from tangent line}\\ \text{at } (0,P(0))}}$$

$$\begin{aligned} P_{\text{approx}}(1) &= P_{\text{approx}}(0.5) + P'_{\text{approx}}(0.5)\cdot(t-0.5)\Big|_{t=1} \\ &= 1.75 \qquad + (-0.72)\cdot(t-0.5)\Big|_{t=1} \qquad = \underbrace{1.39}_{\substack{\text{Estimate from tangent line}\\ \text{at } (0.5,P_{\text{approx}}(0.5))}} \end{aligned}$$



This process becomes more accurate the more (approximate) tangent lines we use. Here is a picture showing what happens if we use four tangent lines.



The process of using many tangent lines to iteratively approximate a solution to a differential equation is called *Euler's method*, and the resulting approximation is called an *Euler approximation*.

Euler's Method. Let $y'(t) = f(t,y)$ be a differential and let y be a solution satisfying $y(t_0) = y_1$. The ***Euler approximation to y with step size Δ*** is the sequence of points (t_0, y_0), (t_1, y_2), ..., where

$$\begin{aligned} y_1 &= y_0 + \Delta\cdot f(t_0,y_0) & t_1 &= t_0 + \Delta \\ y_2 &= y_1 + \Delta\cdot f(t_1,y_1) & t_2 &= t_1 + \Delta \\ &\vdots & &\vdots \\ y_n &= y_{n-1} + \Delta\cdot f(t_{n-1},y_{n-1}) & t_n &= t_{n-1} + \Delta \end{aligned}$$

When applying Euler's method, the big decision is how many tangent lines to use. This is often expressed in terms of a *step size*, typically denoted by Δ, where $\Delta = \frac{\text{domain of approximation}}{\text{\# of tangent lines used}}$.

Example. Use Euler's method to approximate $P(1)$ for the differential equation

$$P'(t) = \sin\Big(\frac{\pi}{2}\cdot P(t) - t\Big) - 1.5$$

with initial condition $P(0) = 3$, using a step size of $\Delta = .25$.

We will implement Euler's method by making a table to help us track the relevant quantities.

Module 2 — Simulation

Step	t	$P_{\text{approx}}(t)$	$P'_{\text{approx}}(t)$
⋮	⋮	⋮	⋮

We can start by filling out the "Step" and t columns. We start at $t = 0$ and at Step 0. Since we have a step size of $\Delta = .25$, we will increment t by Δ at each step until we reach $t = 1$.

Step	t	$P_{\text{approx}}(t)$	$P'_{\text{approx}}(t)$
0	0		
1	0.25		
2	0.5		
3	0.75		
4	1		

At time $t = 0$, we know the exact value of $P(t)$ and of $P'(t)$, so we can fill in the first row of the table.

Step	t	$P_{\text{approx}}(t)$	$P'_{\text{approx}}(t)$
0	0	3	-2.5
1	0.25		
2	0.5		
3	0.75		
4	1		

To find $P_{\text{approx}}(0.25)$, we use a tangent line centered at $(t, P) = (0, 3)$. Thus,

$$P_{\text{approx}}(0.25) = P(0) + P'_{\text{approx}}(0) \cdot (t - 0)\Big|_{t=0.25} = 3 + (-2.5) \cdot (0.25 - 0) = 2.375.$$

We can now fill in the second row of the table, noting that we get $P'_{\text{approx}}(0.25)$ from the formula $P'(t) = \sin(\frac{\pi}{2} \cdot P(t) - t) - 1.5$.

Step	t	$P_{\text{approx}}(t)$	$P'_{\text{approx}}(t)$
0	0	3	-2.5
1	0.25	2.375	-1.833
2	0.5		
3	0.75		
4	1		

We can now compute $P_{\text{approx}}(0.5)$ using a tangent line centered at $(t, P) = (0.25, 2.375)$.

$$\begin{aligned} P_{\text{approx}}(0.5) &= P_{\text{approx}}(0.25) + P'_{\text{approx}}(0.25) \cdot (t - 0.25)\Big|_{t=0.5} \\ &= 2.375 + (-1.833) \cdot (0.5 - 0.25) = 1.917. \end{aligned}$$

and so

Step	t	$P_{\text{approx}}(t)$	$P'_{\text{approx}}(t)$
0	0	3	-2.5
1	0.25	2.375	-1.833

Step	t	$P_{\text{approx}}(t)$	$P'_{\text{approx}}(t)$
2	0.5	1.917	−0.91
3	0.75		
4	1		

Continuing this process, we can fill in the rest of the table.

Step	t	$P_{\text{approx}}(t)$	$P'_{\text{approx}}(t)$
0	0	3	−2.5
1	0.25	2.375	−1.833
2	0.5	1.917	−0.91
3	0.75	1.689	−0.555
4	1	1.551	

This gives an Euler estimate of

$$P(1) \approx P_{\text{approx}}(1) = 1.551.$$

(Note, there is no need to compute $P'_{\text{approx}}(1)$ since we are not using it, but there is no harm in computing it either.)

Implementing Euler's Method

As seen in the previous example, Euler's method can be efficiently implemented using a table. This processes is worth doing a few times by hand, but, in truth, it's what computers are made for.

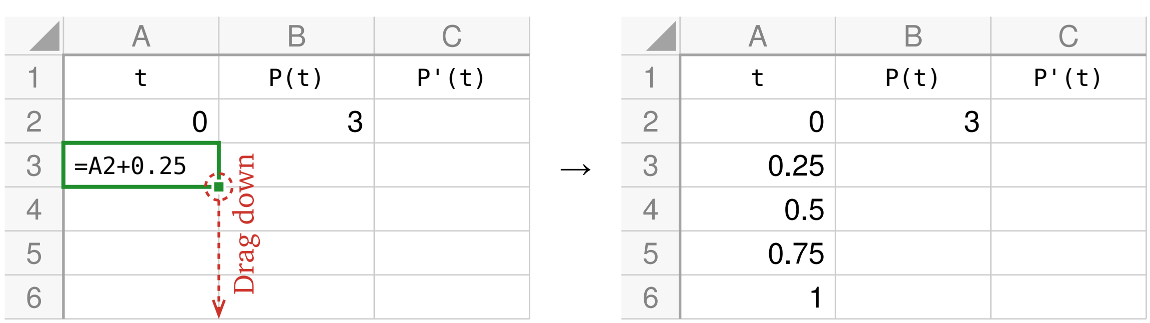
Though you can easily implement Euler's method using general purpose programming languages like Python, Matlab, etc., there is one type of software that excels[5] at implementing table-based algorithms: spreadsheets.

Appendix A gives an overview of how to use spreadsheets. In this section we assume a basic familiarity.

To use a spreadsheet to implement Euler's method, we will recreate the table from the previous example. We start by

- labelling our columns
- inputting our initial conditions, and
- setting up a formula that increments t by Δ at each step.

For this example, we will use $\Delta = 0.25$ and our familiar initial value problem $P'(t) = \sin(\frac{\pi}{2} \cdot P(t) - t) - 1.5$ with $P(0) = 3$.



[5] Pun intended.

Next, we enter the formula for $P'(t)$, making reference to the appropriate cells to get the values of t and $P(t)$. Here, the formula we enter into `C2` is `=SIN(PI() / 2 * B2 - A2) - 1.5`.[6]

	A	B	C
1	t	P(t)	P'(t)
2	0	3	=SIN(PI()
3	0.25		
4	0.5		
5	0.75		
6	1		

Drag down

→

	A	B	C
1	t	P(t)	P'(t)
2	0	3	–2.5
3	0.25		–1.7474
4	0.5		–1.97943
5	0.75		–2.18164
6	1		–2.34147

Note: our spreadsheet currently has *incorrect* values for P' since the formula for $P'(t)$ in the spreadsheet is referencing cells that have not yet been populated with values.[7]

Finally, we can enter the formula for $P(t)$. Based on the tangent line approximation, we enter in the cell `B3` the formula `=B2+0.25*C2` (make sure you know where this formula comes from before continuing).

	A	B	C
1	t	P(t)	P'(t)
2	0	3	–2.5
3	0.25	=B2+0.25*C	–1.7474
4	0.5		–1.97943
5	0.75		–2.18164
6	1		–2.34147

→

	A	B	C
1	t	P(t)	P'(t)
2	0	3	–2.5
3	0.25	2.375	–1.83259
4	0.5	1.91685	–0.91036
5	0.75	1.68926	–0.55483
6	1	1.55055	–0.50912

Our spreadsheet has now taken care of all the tedious calculations! It is also *reactive*. For example, if we change the initial conditions (i.e., change the value of $P(0)$ in cell `B2`), the spreadsheet will automatically recompute the value in all other cells.

Accuracy of Euler's Method

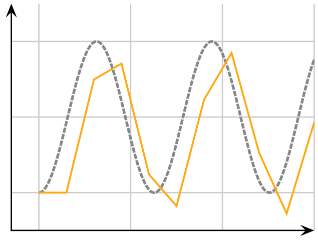
When using Euler's method, in general, smaller step sizes produce more accurate the approximation. And, when taking a limit towards an infinitely small step size, the approximation converges to an exact solution.

For example, consider the initial value problem $y'(t) = \sin(5t)$ with $y(0) = 0.8$. We can solve this exactly to get $y(t) = 1 - 0.2\cos(5t)$. Comparing the exact solution to Euler approximations with different step sizes, we see that the smaller the step size, the more accurate the approximation.

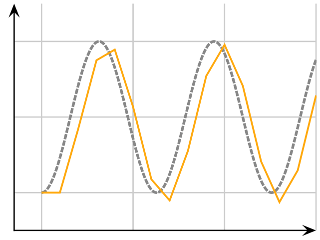
[6] To get the value π we must enter `PI()` with the parenthesis `()` at the end.

[7] Spreadsheets typically interpret empty cells as the number 0. They do this instead of producing an error message when you use an empty cell in a formula.

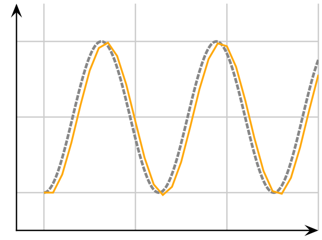
Exact Solution vs. Euler Approx ($\Delta = 0.3$)



Exact Solution vs. Euler Approx ($\Delta = 0.2$)



Exact Solution vs. Euler Approx ($\Delta = 0.1$)



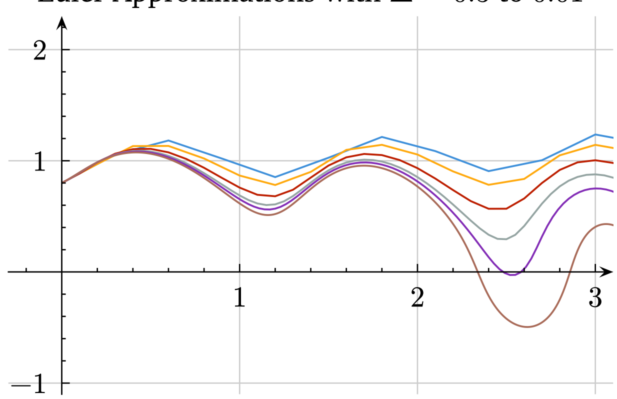
In the above example, we knew the exact solution, but what if we didn't? It is possible to get explicit bounds on the error of an Euler approximation through Taylor's Remainder Theorem and a careful analysis of Euler's method[8], but we will take a more experimental approach.

Consider the initial value problem

$$y' = \frac{\sin(5x + y)}{.2 + y^2} \qquad \text{and} \qquad y(0) = 0.8.$$

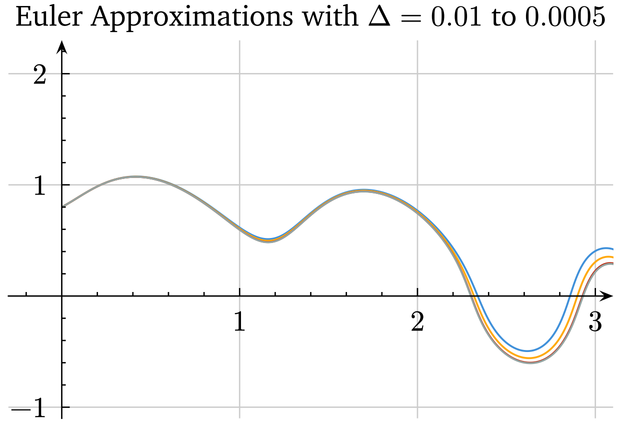
We don't have an explicit solution to this initial value problem, but we can plot Euler approximations with different step sizes and compare them. Plotting with $\Delta = 0.3, 0.2, 0.1, 0.05, 0.03,$ and 0.01 we see different Δ's result in very different curves. What does the exact solution look like?

Euler Approximations with $\Delta = 0.3$ to 0.01



Initially, it may not seem like the approximations are converging to an exact solution. However, if we keep plotting with smaller and smaller Δ's, we see that the approximations start to settle down to a consistent shape.

Euler Approximations with $\Delta = 0.01$ to 0.0005



[8] See https://math.libretexts.org/Courses/Monroe_Community_College/MTH_225_Differential_Equations/03%3A_Numerical_Methods/3.01%3A_Euler%27s_Method for a detailed exposition.

When approximations start looking consistent, this provides strong evidence that the approximations are close to the exact solution. Of course, this evidence is not as strong as a mathematical *proof*, but it is usually sufficient for doing science (where there are likely other sources of error far greater than the error arising from Euler's method).

> **Example.** Use Euler's method to estimate whether the solution to the initial value problem … is periodic or not
>
> XXX Finish. Come up with a nice example where it's not super obvious.

Practice Problems

1. Explain what you need to do in two different ways.
   (a) Way 1
   (b) Way 2
2. A second question.
3. A third question.

Solutions for Module 2

1. (a) Way 1 is good
   (b) Way 2 is better
2. The answer to the second question.
3.

10 Consider the following argument for the population model **S** where $P'(t) = P(t) \cdot |\sin(2\pi t)|$ with $P(0) = 10$:

At $t = 0$, the change in population $\approx P'(0) = 0$, so

$$P(1) \approx P(0) + P'(0) \cdot 1 = P(0) = 10.$$

At $t = 1$, the change in population $\approx P'(1) = 0$, so

$$P(2) \approx P(1) + P'(1) \cdot 1 = P(0) = 10.$$

And so on.

So, the population of starfish remains constant.

10.1 Do you believe this argument? Can it be improved?

10.2 Simulate an improved version using a spreadsheet.

11 (Simulating $\mathbf{M}_{\infty}$ from Core Exercise 6 with different Δs)

Time	Pop. ($\Delta = 0.1$)	Time	Pop. ($\Delta = 0.2$)
0.0	10	0.0	10
0.1	11.1	0.2	12.2
0.2	12.321	0.4	14.884
0.3	13.67631	0.6	18.15848
0.4	15.1807041	0.8	22.1533456

11.1 Compare $\Delta = 0.1$ and $\Delta = 0.2$. Which approximation grows faster?

11.2 Graph the population estimates for $\Delta = 0.1$ and $\Delta = 0.2$ on the same plot. What does the graph show?

11.3 What Δs give the largest estimate for the population at time t?

11.4 Is there a limit as $\Delta \to 0$?

12 Consider the following models for starfish growth:

M # new children per year $\sim$ current population.
N # new children per year $\sim$ current population times resources available per individual.
O # new children per year $\sim$ current population times the fraction of total resources remaining.

12.1 Guess what the population vs. time curves look like for each model.

12.2 Create a differential equation for each model.

12.3 Simulate population vs. time curves for each model (but pick a common initial population).

13 Recall the models

- **M** # new children per year $\sim$ current population.
- **N** # new children per year $\sim$ current population times resources available per individual.
- **O** # new children per year $\sim$ current population times the fraction of total resources remaining.

13.1 Determine which population grows fastest in the short term and which grows fastest in the long term.

13.2 Are some models more sensitive to your choice of Δ when simulating?

13.3 Are your simulations for each model consistently underestimates? Overestimates?

13.4 Compare your simulated results with your guesses from question What did you guess correctly? Where were you off the mark?

Systems and Models with Interacting Terms

In this module you will learn

- How to build models using systems of differential equations.
- How to approximate solutions to systems of differential equations.
- How to graph solutions to systems of differential equations in component space and phase space.

Modelling

In the previous two modules, we modeled a single quantity. But, what happens when we have multiple interrelated quantities?

To explore this, we will model populations of *Yellow Meadow Ants*. The Yellow Meadow Ant is a species of *farming* ant. They tend to farms of aphids, which are small insects that suck the sap from plants. The ants protect the aphids from predators and in return the aphids secrete a sugary substance called honeydew, which the ants eat. The population of ants and the population of aphids are symbiotically interrelated, as the growth of one population depends on the other.

We can create a *system* of differential equations to model the interrelated populations:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\#\text{ ants at minute } t) = a(\#\text{ ants at minute } t) + b(\#\text{ aphids at time } t)$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(\#\text{ aphids at minute } t) = c(\#\text{ ants at minute } t) - d(\#\text{ aphids at time } t)$$

The first equation models the growth of the ant population, with a representing the natural growth of ants in the absence of aphids and b representing the growth of the ant population due to the presence of aphids (this term provides a “boost” to the ant population since the presence of aphids will mean more food for the ants).

The second equation models the growth of the aphid population, with c representing the growth of the aphid population due to the presence of ants (this term has a positive effect on the growth of aphids since they will be protected from predators), and d representing the natural death of aphids in the absence of ants (they will be eaten by other creatures if ants aren’t around to protect them).

The parameters $a, b, c, d \geq 0$ depend on the habitat. They could be from data, but we will instead use artificially nice values of a, b, c, d.

Simulation

We will use a modified version of Euler’s method (see Module 2) to simulate a solution to a system of differential equations. The change will be that we will use two tangent lines to estimate the next data point—one tangent line for ants and one for aphids.

For now, we will assume $a = 1, b = c = d = \frac{1}{2}$ and that units are in thousands (or ands and aphids). The initial population will be 10 thousand ants and 100 thousand aphids. Using a time step of $\Delta = 0.25$, we compute

$$(\#\text{ Ants})'(0) = 1 \cdot 10 + \frac{1}{2} \cdot 100 = 60$$
$$(\#\text{ Aphids})'(0) = \frac{1}{2} \cdot 10 - \frac{1}{2} \cdot 100 = -45$$

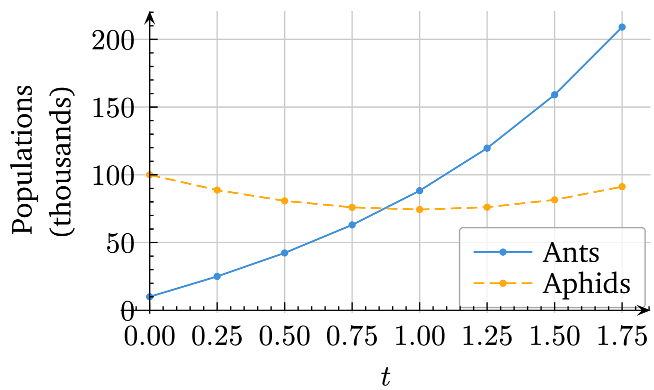
and so

$$(\#\ \text{Ants})(0.25) \approx 10 + 0.25 \cdot 60 = 25$$
$$(\#\ \text{Aphids})(0.25) \approx 100 + 0.25 \cdot (-45) = 88.75$$

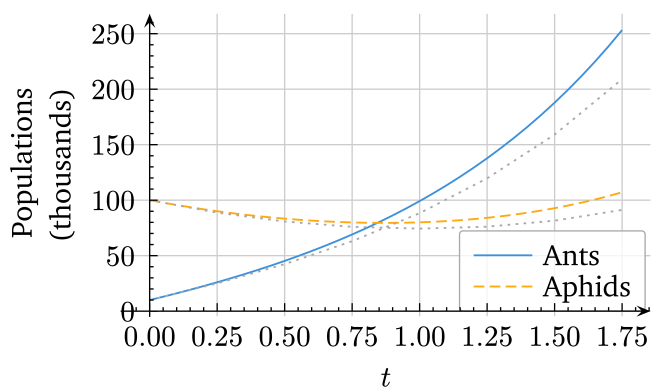
We can now repeat this processes at $t = 0.25$ to find approximate values for the number of ants and aphids at $t = 0.5$, etc.. Repeating until $t = 1.75$, we arrive at the following table of values:

Time	(approximate) # Ants	(approximate) # Aphids
0	10	100
0.25	25	88.75
0.5	42.344	80.781
0.75	63.027	75.977
1	88.281	74.358
1.25	119.646	76.098
1.5	159.07	81.542
1.75	209.03	91.233

In the graph below, our simulated solution is shown with the solid curve representing the population of ants and the dashed curve representing the population of aphids.



Of course, if we wanted a more accurate simulation, we could use a smaller step size. Below is a graph using a step size of $\Delta = 0.05$, The new, more accurate estimates are shown (solid and dashed) along with the old estimates (dotted).

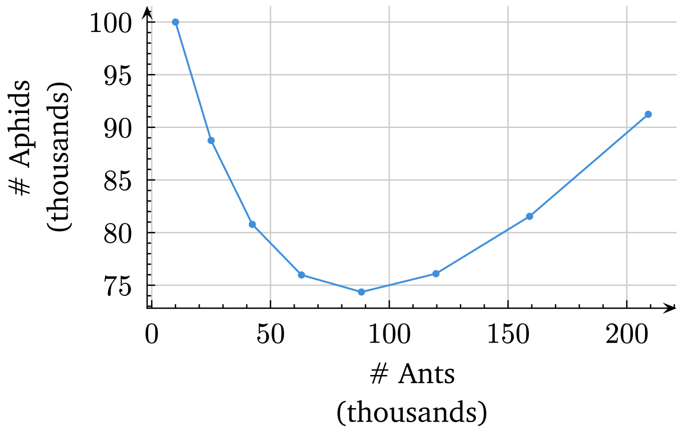


Component and Phase Spaces

The graphs above is called *component graphs*. They show the dependent variables (the populations of ants and aphids) vs. the independent variable (time).[9]

[9]It is actually two component graphs, one for the ants and one for the aphids.

However, we often want to consider the relationship *between the dependent variables*. In this example, we might plot the population of ants vs. the population of aphids.



This plot suggests a relationship: there is a threshold where if the # ants is above that threshold, they enable growth in # aphids. More analysis is needed to see if this observation is valid, but the graph points us in the right direction.[10]

Plots like the one above are called *phase plots* or *plots in phase space*. The space where each axis corresponds to a dependent variable is called *phase space* or the *phase plane*.

Component Graph & Phase Plane. For a differential equation involving the functions F_1, F_2, ..., F_n, and the variable t, the ***component graphs*** are the n graphs of $(t, F_1(t))$, $(t, F_2(t))$, ...

The ***phase plane*** or ***phase space*** associated with the differential equation is the n-dimensional space with axes corresponding to the values of F_1, F_2, ..., F_n.

XXX We need a nice conclusion

Example. The Three-dimensional Lorenz Equations

Phase space is not limited to two dimensions. Consider the Lorentz equations, introduced by Edward Lorenz to demonstrate the inherent challenge in weather prediction.[11] The Lorenz equations are

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sigma(y - x)$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = x(\rho - z) - y$$
$$\frac{\mathrm{d}z}{\mathrm{d}t} = xy - \beta z$$

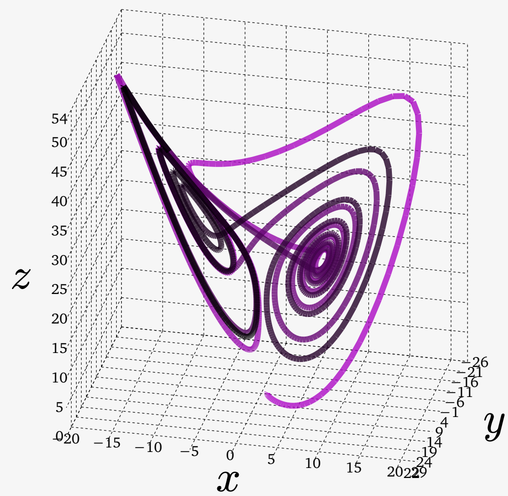
where $\sigma = 10$, $\rho = 28$, and $\beta = \frac{8}{3}$.

Since there are three dependent variables (x, y, and z), the phase space associated with the Lorenz equations is three dimensional.

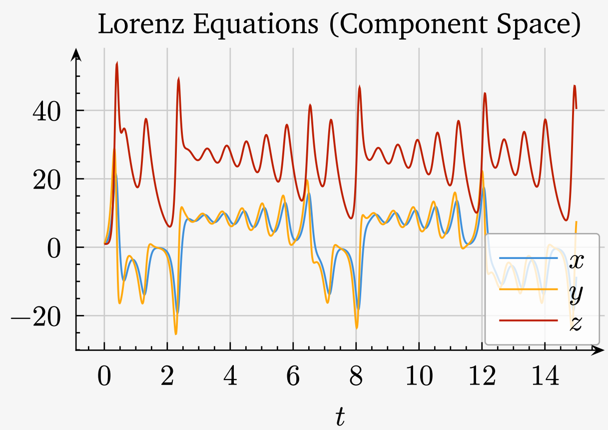
Simulating using Euler's method, we get the following phase-space plot.

[10] See if you can find what the threshold # ants is by analyzing the differential equation directly.

[11] The Lorenz equations would go on to become a foundational example in the study of chaos theory—a deterministic but hard to predict dependence on initial conditions.



From the plot in phase space, we can see the spiralling nature of solutions. Something that is much harder to see from the component graphs.



Practice Problems

1. Explain what you need to do in two different ways.
   (a) Way 1
   (b) Way 2
2. A second question.
3. A third question.

Solutions for Module 3

1. (a) Way 1 is good
   (b) Way 2 is better
2. The answer to the second question.
3.

14 A simple model for population growth has the form

$$P'(t) = b \cdot P(t)$$

where b is the birth rate.

14.1 Create a better model for population that includes both births and deaths.

15 *Lotka-Volterra Predator-Prey* models predict two populations, F (foxes) and R (rabbits), simultaneously. They take the form

$$F'(t) = (B_F - D_F) \cdot F(t)$$
$$R'(t) = (B_R - D_R) \cdot R(t)$$

where $B_{\{?\}}$ stands for births and $D_{\{?\}}$ stands for deaths.

We will assume:

($\mathrm{P}_{\text{foxes 1}}$) Foxes die at a constant rate.
($\mathrm{P}_{\text{foxes 2}}$) Foxes mate when food is plentiful.
($\mathrm{P}_{\text{rabbits}}$) Rabbits mate at a constant rate.
($\mathrm{P}_{\text{predation}}$) Foxes eat rabbits.

15.1 Speculate on when B_F, D_F, B_R, and D_R would be at their maximum(s)/minimum(s), given our assumptions.

15.2 Come up with appropriate formulas for B_F, B_R, D_F, and D_R.

16 Suppose the population of F (foxes) and R (rabbits) evolves over time following the rule

$$F'(t) = (0.01 \cdot R(t) - 1.1) \cdot F(t)$$
$$R'(t) = (1.1 - 0.1 \cdot F(t)) \cdot R(t)$$

16.1 Simulate the population of foxes and rabbits with a spreadsheet.

16.2 Do the populations continue to grow/shrink forever? Are they cyclic?

16.3 Should the humps/valleys in the rabbit and fox populations be in phase? Out of phase?

17 Open the spreadsheet

`https://uoft.me/foxes-and-rabbits`

which contains an Euler approximation for the Foxes and Rabbits population.

$$F'(t) = (0.01 \cdot R(t) - 1.1) \cdot F(t)$$
$$R'(t) = (1.1 - 0.1 \cdot F(t)) \cdot R(t)$$

17.1 Is the maximum population of the rabbits over/under estimated? Sometimes over, sometimes under?

17.2 What about the foxes?

17.3 What about the min populations?

18 Open the spreadsheet

`https://uoft.me/foxes-and-rabbits`

which contains an Euler approximation for the Foxes and Rabbits population.

$$F'(t) = (0.01 \cdot R(t) - 1.1) \cdot F(t)$$
$$R'(t) = (1.1 - 0.1 \cdot F(t)) \cdot R(t)$$

Component Graph & Phase Plane. For a differential equation involving the functions F_1, F_2, …, F_n, and the variable t, the ***component graphs*** are the n graphs of $(t, F_1(t))$, $(t, F_2(t))$, ….

The ***phase plane*** or ***phase space*** associated with the differential equation is the n-dimensional space with axes corresponding to the values of F_1, F_2, …, F_n.

18.1 Plot the Fox vs. Rabbit population in the phase plane.

18.2 Should your plot show a closed curve or a spiral?

18.3 What "direction" do points move along the curve as time increases? Justify by referring to the model.

18.4 What is easier to see from plots in the phase plane than from component graphs (the graphs of fox and rabbit population vs. time)?

Equilibrium and Long-Term Behaviour

In this module you will learn

- What equilibrium solutions and equilibrium points are.
- How to equilibrium points related to the long term behavior of a differential equation/ system of differential equations.

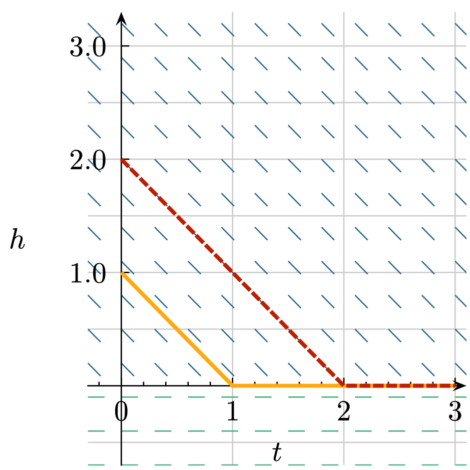
Models built from differential equations will tell you how quantities *change*. Equally important, though, are the conditions under which a quantity *does not* change. In science and engineering, when a when a system is in a state that doesn't change, we say it is at *equilibrium*. In the context of differential equations, solutions that don't change are called *equilibrium solutions*.

Equilibrium Solution. An ***equilibrium solution*** to a differential equation or system of differential equations is a solution that is constant in the independent variable(s).

Suppose you are modelling a leaf falling from a balcony onto the ground. Let $h(t)$ represent the height of the leaf above the ground at time t. If we assume that air resistance causes the leaf to fall at a constant speed of 1 m/s, we can set up a differential equation to model the leaf's motion.

$$h' = \begin{cases} -1 & \text{if} \quad h > 0 \\ 0 & \text{if} \quad h \leq 0 \end{cases}$$

Looking at a slope field for this equation, we can see that most solutions look "L"-shaped, first decreasing to zero and then remaining constant.



But, there is one solution that is qualitatively different: $h(t) = 0$. Indeed, the function $h(t) = 0$ satisfies $h'(t) = 0$, and so is a solution to the differential equation. It is the *equilibrium solution* to the differential equation corresponding to the leaf resting on the ground (not actually falling at all).

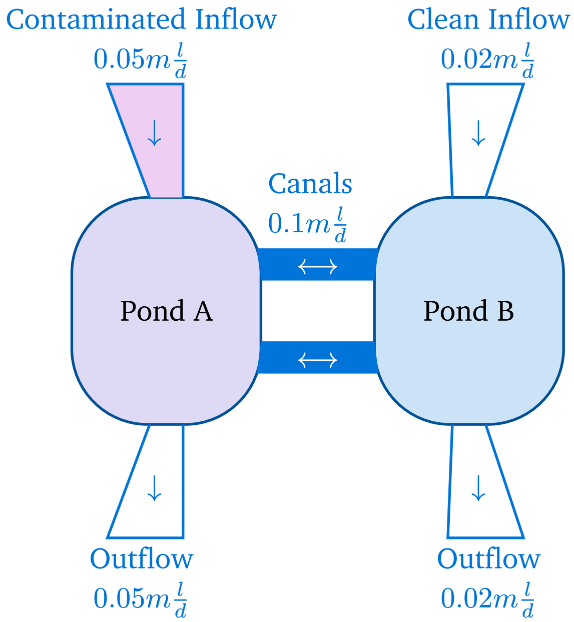
Let's consider a more complicated example.

Two ponds, Pond A and Pond B, each with a volume of 1 million litres, are connected by canals. Pond A is fed by a stream with contaminated with pesticides from a nearby farm. Pond B is fed by a mountain stream of clean water. The ponds exchange water through their canals at a rate of 0.1 million litres per day. Additionally, the ponds have spill-gates that allow any excess water to flow out of each pond so they maintain a constant volume.

We will assume:

- The ponds are well-mixed.

- Contaminated water flows into Pond A at a rate of 0.05 million litres per day with 1 kilogram of pesticide per million litres.
- Clean water flows into Pond B at a rate of 0.02 million litres per day.

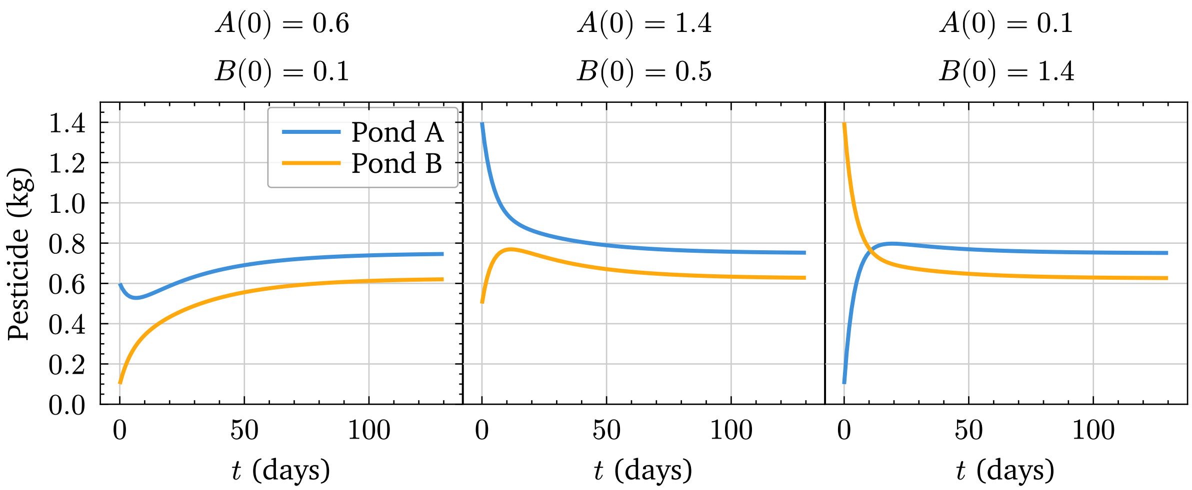


Let $A(t)$ be the amount of pesticide in Pond A at time t, and let $B(t)$ be the amount of pesticide in Pond B at time t.

We can model A and B with the following system of differential equations.[12]

$$\begin{aligned} A' &= -0.15 \cdot A + 0.1 \cdot B + 0.05 \\ B' &= 0.1 \cdot A - 0.12 \cdot B \end{aligned}$$

With our model defined, we can now make plots showing the amount of pesticide vs. time given different initial conditions.



Using our physical intuition, we would expect that the amount of pesticide in each pond tends towards a constant (i.e., an equilibrium), and that's exactly what we see in the plots. No matter

[12]To come up with this model, notice that A', the change in the amount of pesticide in Pond A, is equal to the inflow of pesticide minus the outflow of pesticide. Pond A has an inflow of 0.05 kg / day from farm runoff and an inflow of $0.1 \cdot B(t)$ kg / day from Pond B. It has an outflow of $0.1 \cdot A(t)$ kg / day to Pond B and an outflow of $0.05 \cdot A(t)$ kg / day to the environment. Thus $A' = 0.05 + 0.1 \cdot B - 0.1 \cdot A - 0.05 \cdot A = -0.15 \cdot A + 0.1 \cdot B + 0.05$. A similar argument will produce an equation for B'.

the initial conditions, the eventual amount of pesticide in Pond A is slightly less than 0.8 kg and the eventual amount of pesticide in Pond B is slightly more than 0.6 kg.

We can compute the equilibrium solution to this differential equation exactly. Since we know that an equilibrium solution is constant, we know that the derivative of an equilibrium solution is always zero. Solving

$$\begin{aligned} 0 &= A' = -0.15 \cdot A + 0.1 \cdot B + 0.05 \\ 0 &= B' = 0.1 \cdot A - 0.12 \cdot B \end{aligned}$$

we arrive at the unique solution $(A, B) = (0.75, 0.625)$. In other words,

$$\begin{aligned} A(t) &= 0.75 \\ B(t) &= 0.625 \end{aligned}$$

is the only equilibrium solution to this system.

Example. Differential equations may have more than one equilibrium solution. Find all equilibrium solutions to

$$\begin{aligned} P' &= P \cdot Q - 4P + 3 \\ Q' &= 2P - Q \cdot (P + 1) \end{aligned}$$

To find the equilibrium solutions, we set $P' = 0$ and $Q' = 0$ and solve the resulting system of equations.

$$\begin{aligned} 0 &= P \cdot Q - 4P + 3 \\ 0 &= 2P - Q \cdot (P + 1) \end{aligned}$$

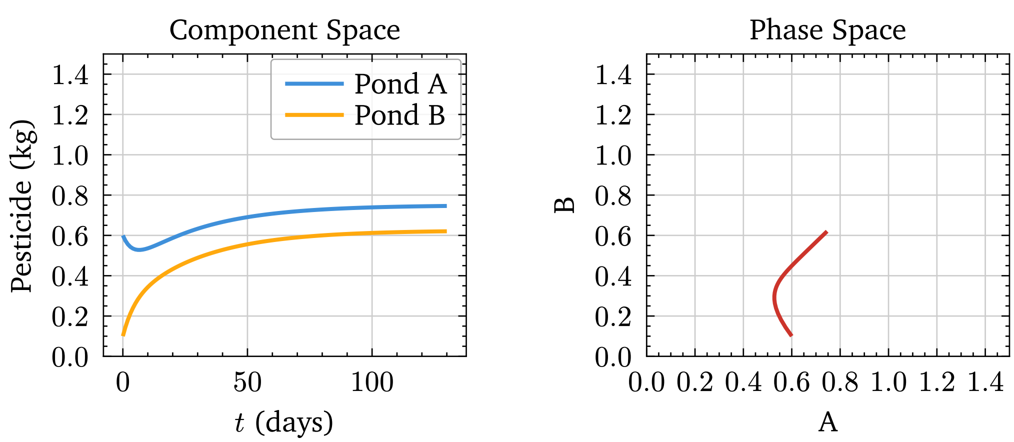
implies either $(P, Q) = (1, 1)$ or $(P, Q) = \left(-\frac{3}{2}, 6\right)$.

Therefore, the equilibrium solutions are

$$\begin{aligned} P(t) &= 1 \\ Q(t) &= 1 \end{aligned} \quad \text{and} \quad \begin{aligned} P(t) &= -\frac{3}{2} \\ Q(t) &= 6 \end{aligned}$$

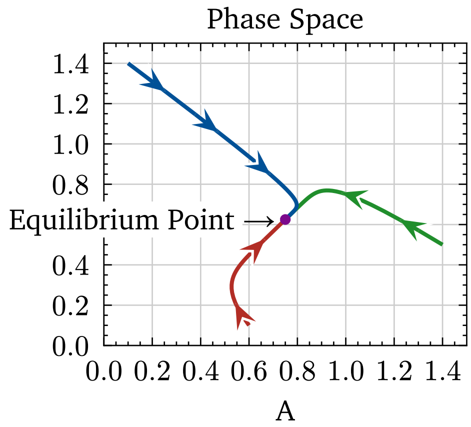
Equilibrium Points

We can also examine equilibrium solutions in phase space. Using our pond example, we can make a plot in phase space.



As we noticed, all solutions tend towards the equilibrium solutions $A(t) = 0.75$ and $B(t) = 0.625$. In phase space, this manifests as all solutions tending towards the point $(A, B) =$

$(0.75, 0.625)$. Further, if we graph the equilibrium solution $A(t) = 0.75$ and $B(t) = 0.625$ in phase space, instead of a curve, we get a single point $(A, B) = (0.75, 0.625)$. We call this point an *equilibrium point*.



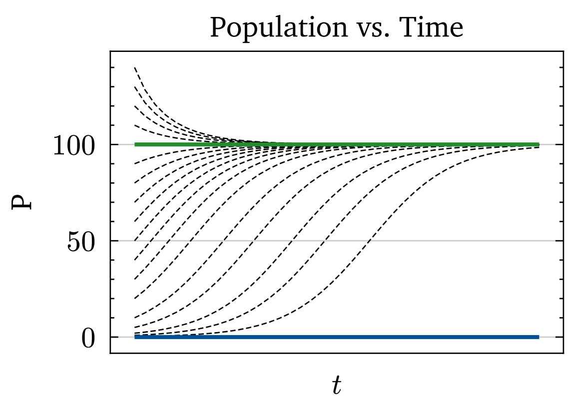
Types of Equilibrium Solutions

In the mixing ponds example, all solutions tended towards the equilibrium solution (the equilibrium solution is *attracting*). This isn't always the case.

In fact, consider the following population model based on the assumption of exponential growth and a carrying capacity[13]:

$$P'(t) = P(t) \cdot (100 - P(t))$$

It has two equilibrium solutions: $P(t) = 0$ and $P(t) = 100$ (shown in solid colors). All other solutions are non-constant (dashed lines).



We call the equilibrium solution $P(t) = 0$ *unstable* and *repelling*. That is, while it is true that a population of exactly zero will stay that way, if there is even one individual (in this model at least), the population will grow substantially. Alternatively, we call the equilibrium solution $P(t) = 100$ *stable* and *attracting*. If the population is exactly 100, it will stay that way, but if the population is slightly less than or slightly more than 100, it will tend towards 100.

In general, equilibrium solutions can be classified as attracting, repelling, stable, and/or unstable depending on the behaviour of solutions near that equilibrium (i.e., solutions with initial conditions near the equilibrium point).

[13] 100 represents the carrying capacity of the environment. If the population exceeds the carrying capacity, individuals start dying.

Classification of Equilibria. An equilibrium solution f is called

- ***attracting*** if locally, solutions converge to f;
- ***repelling*** if there is a fixed distance so that locally, solutions tend away from f by that fixed distance;
- ***stable*** if for any fixed distance, locally, solutions stay within that fixed distance of f; and,
- ***unstable*** if f is not stable.

The above definition uses the term *local* to refer to solutions that passes through points "close to" that of the equilibrium solution. This can be made precise using ε-δ definitions.

Classification of Equilibria (Formal). An equilibrium solution f is called

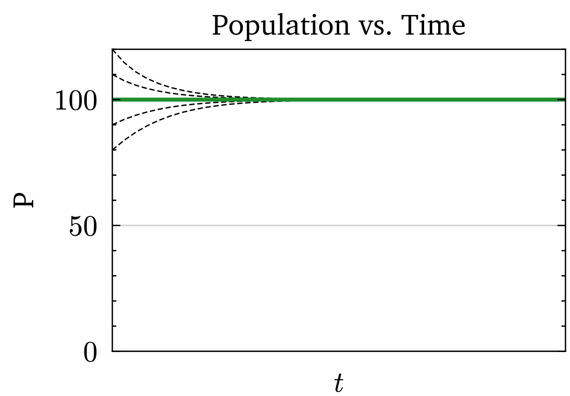
- ***attracting at time*** t_0 if there exists $\varepsilon > 0$ such that for all solutions g satisfying $|g(t_0) - f(t_0)| < \varepsilon$, we have $\lim_{t\to\infty} f(t) = \lim_{t\to\infty} g(t)$.
- ***repelling at time*** t_0 if there exists $\varepsilon > 0$ and $\delta > 0$ such that for all solutions g that satisfy $0 < |g(t_0) - f(t_0)| < \varepsilon$ there exists $T \in \mathbb{R}$ so that for all $t > T$ we have $|g(t) - f(t)| > \delta$.
- ***stable at time*** t_0 if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that for all solutions g satisfying $|g(t_0) - f(t_0)| < \delta$ we have $|g(t) - f(t)| < \varepsilon$ for all $t > t_0$.
- ***unstable at time*** t_0 if f is not stable at time t_0.

f is called attracting/repelling/stable/unstable if it has the corresponding property for all t.

Whether using the formal or informal definition, the important thing is to have an intuition about what different types of equilibrium solutions look like, both in *component space* and *phase space*.

Stable and Attracting

We've already seen that for $P' = P \cdot (100 - P)$, the equilibrium solution $P(t) = 100$ is stable and attracting.

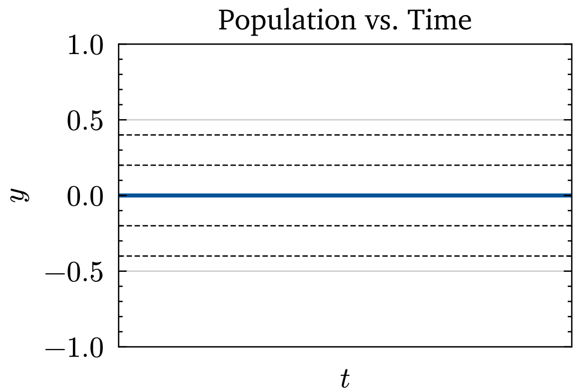


- It is *stable* because if a solution P^* (dashed curves above) starts close to $P(t) = 100$, then it will stay close to $P(t) = 100$.
- The equilibrium solution $P(t) = 100$ is *attracting* because if a solution P^* (dashed curves above) starts close to $P(t) = 100$, its limit as $t \to +\infty$ will actually be 100.

Stable and not Attracting

Consider the differential equation $y' = 0$. This equation has solutions of the form $y(t) = k$, where k is a constant. These are all equilibrium solutions!

Let's focus on the equilibrium solution $y(t) = 0$.



- This solution is *stable* because if a solution $y^*(t) = k$ (dashed curve above) starts close to $y(t) = 0$, it will stay close to $y(t) = 0$; in fact

its distance from $y(t)$ will never change.

- However, $y(t) = 0$ is *not attracting*, because $y^*(t) = k \quad \not\to \quad 0$.

Unstable and Repelling

XXX Finish

Unstable and not Repelling

XXX Finish

Other Classifications

There cannot be unstable and attracting equilibria, nor can there be stable and repelling equilibria. But there can be other behaviours.

Consider the Van der Pol system:

$$x' = y$$
$$y' = \mu \cdot (1 - x^2) \cdot y - x$$

for $\mu > 0$.

Solutions to this system all end up with oscillatory behaviour and they all have the same period. Here solutions are not attracted to an equilibrium, but they are attracted towards a solution which is perfectly periodic (in this situation we say that the system has a *limit cycle*). The study of what behaviour solutions can be "attracted to" leads to concepts like fractals and chaos, and may be studied in an advanced differential equations course or a course on dynamical systems.

XXX Figure

Example. Find an classify all equilibrium solutions to XXX Finish

XXX Finish

Practice Problems

1. Explain what you need to do in two different ways.
   (a) Way 1
   (b) Way 2
2. A second question.
3. A third question.

Solutions for Module 4

1. (a) Way 1 is good
   (b) Way 2 is better
2. The answer to the second question.
3.

19 Open the spreadsheet

`https://uoft.me/foxes-and-rabbits`

which contains an Euler approximation for the Foxes and Rabbits population.

$$F'(t) = (0.01 \cdot R(t) - 1.1) \cdot F(t)$$
$$R'(t) = (1.1 - 0.1 \cdot F(t)) \cdot R(t)$$

Equilibrium Solution. An ***equilibrium solution*** to a differential equation or system of differential equations is a solution that is constant in the independent variable(s).

19.1 By changing initial conditions, what is the "smallest" curve you can get in the phase plane? What happens at those initial conditions?

19.2 What should F' and R' be if F and R are *equilibrium solutions*?

19.3 How many equilibrium solutions are there for the fox-and-rabbit system? Justify your answer.

19.4 What do the equilibrium solutions look like in the phase plane? What about their component graphs?

20 Recall the logistic model for starfish growth (introduced in Core Exercise 12):

O # new children per year $\sim$ current population times the fraction of total resources remaining

which can be modeled with the equation

$$P'(t) = k \cdot P(t) \cdot \left(1 - \frac{R_i}{R} \cdot P(t)\right)$$

where

- $P(t)$ is the population at time t
- k is a constant of proportionality
- R is the total number of resources
- R_i is the resources that one starfish wants to consume

Use $k = 1.1$, $R = 1$, and $R_i = 0.1$ unless instructed otherwise.

20.1 What are the equilibrium solutions for model **O**?

20.2 What does a "phase plane" for model **O** look like? What do graphs of equilibrium solutions look like?

20.3 Classify the behaviour of solutions that lie *between* the equilibrium solutions. E.g., are they increasing, decreasing, oscillating?

21

Classification of Equilibria. An equilibrium solution f is called

- *attracting* if locally, solutions converge to f;
- *repelling* if there is a fixed distance so that locally, solutions tend away from f by that fixed distance;
- *stable* if for any fixed distance, locally, solutions stay within that fixed distance of f; and,
- *unstable* if f is not stable.

Let

$$F'(t) = ?$$

be an unknown differential equation with equilibrium solution $f(t) = 1$.

21.1 Draw an example of what solutions might look like if f is *attracting*.

21.2 Draw an example of what solutions might look like if f is *repelling*.

21.3 Draw an example of what solutions might look like if f is *stable*.

21.4 Could f be stable but *not* attracting?

22 Recall the starfish population model **O** given by

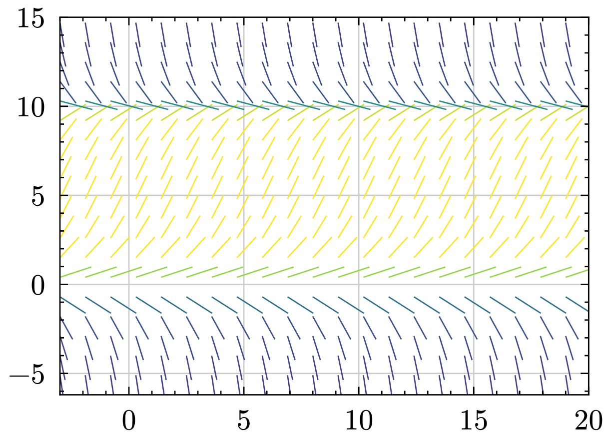
$$P'(t) = k \cdot P(t) \cdot \left(1 - \frac{R_i}{R} \cdot P(t)\right)$$

Use $k = 1.1$, $R = 1$, and $R_i = 0.1$ unless instructed otherwise.

22.1 Classify the equilibrium solutions for model **O** as attracting, repelling, stable, unstable, or semi-stable.

22.2 Does changing k change the nature of the equilibrium solutions? How can you tell?

23



A *slope field* is a plot of small segments of tangent lines to solutions of a differential equation at different initial conditions.

On the left is a slope field for model **O**, available at

`https://www.desmos.com/calculator/ghavqzqqjn`

23.1 If you were sketching the slope field for model **O** by hand, what line would you sketch (a segment of) at $(5, 3)$? Write an equation for that line.

23.2 How can you recognize equilibrium solutions in a slope field?

23.3 Give qualitative descriptions of different solutions to the *differential equation* used in model **O** (i.e., use words to describe them). Do all of those solutions make sense in terms of *model* **O**?

Qualitative Analysis: Slope Fields and Phase Portraits

In this module you will learn

- How slope fields can be used to analyze a differential equation.
- How phase portraits can be used to analyze a system of differential equations.

Sometimes we are interested in the precise behaviour of a solution to a differential equation, but more often than not, we're interested in the *qualities* of solutions. For example, we may want to know if an equilibrium solution is stable, or if a solution is periodic/initially increasing/unbounded/etc.. Much of this information can be deduced without actually solving or simulating solutions to the differential equation.

Analyzing Slope Fields

In Module 2 we saw *slope fields*.

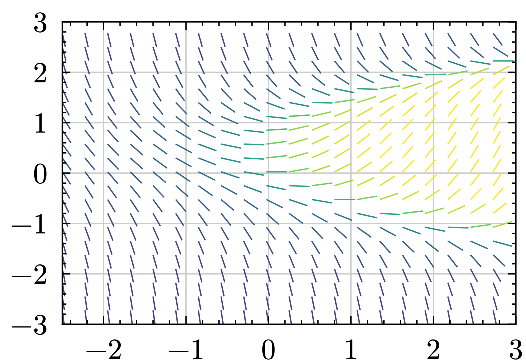
> **Slope Field.** A *slope field* for the differential equation $y' = f(x, y)$ is a collection of line segments that are tangent to different solutions of the differential equation.
>
> The line segments are generated by evaluating $f(x, y)$ over a rectangular grid of points, and at each point (x, y) of the grid creating a line segment of slope $f(x, y)$. The length of the line segments is not important, but should be consistent.

By looking at a slope field, we can visualize what whole ranges of solutions look like.

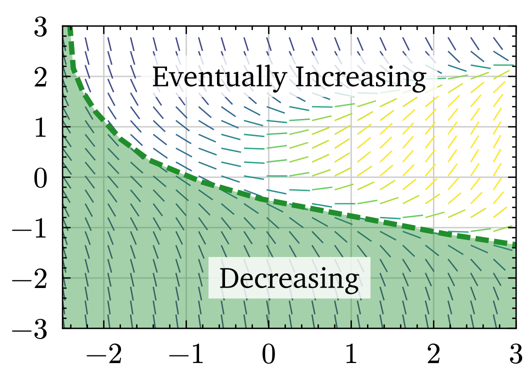
Consider the differential equation

$$y' = y(1 - y) + x$$

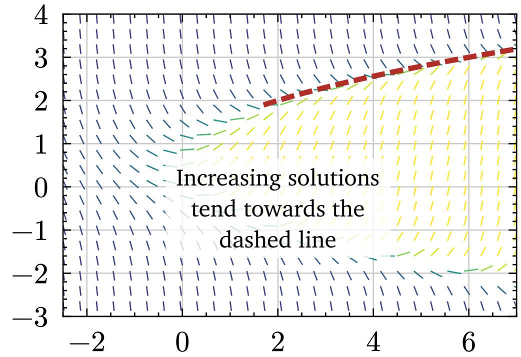
and its slope field below.



From the slope field, we can see there are two qualitatively different types of solutions: those that decrease forever, and those that eventually increase. We can roughly divide the slope field into regions where initial conditions in one region lead to decreasing solutions, and initial conditions in the other region lead to eventually increasing solutions.



We can also see that increasing solutions tend towards a particular increasing solution.

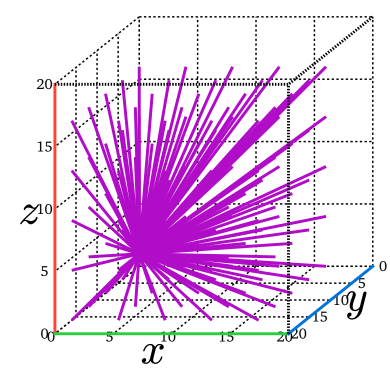


Systems of Differential Equations and Phase Portraits

Consider a modified version of the Ants-and-Aphids model from Module 3, where ants are removed by a farmer at a constant rate (10 thousand per day).

$$\begin{aligned} P'_{\text{ant}} &= P_{\text{ant}} + 0.5 \cdot P_{\text{aphid}} - 10 \\ P'_{\text{aphid}} &= 0.5 \cdot P_{\text{ant}} - 0.5 \cdot P_{\text{aphid}} \end{aligned}$$

How can we make a slope field for this system? If we make slope fields in component space, we miss out on important information. For example, to compute P'_{ant} we need a value of P_{aphid}, but because the number of ants and the number of aphids affect each other, we cannot just choose a value for P_{aphid}. The same issue arises for P'_{aphid}. The conclusion is that a "slope field" for this system must have a P_{ant} axis, a P_{aphid} axis, and a time axis. I.e., it is three-dimensional.



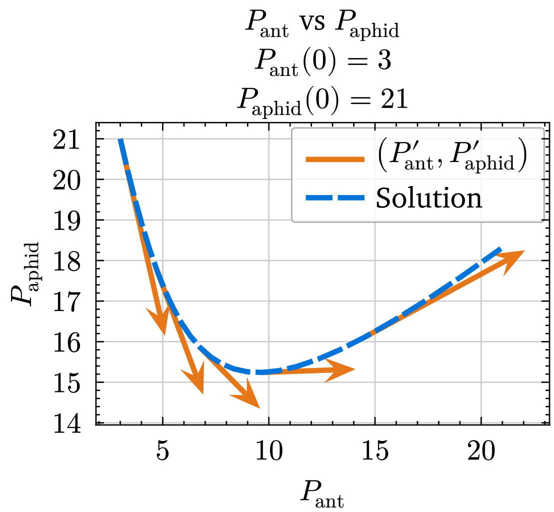
XXX make correct 3d slope field Figure

It's hard to glean information from a three-dimensional slope field. What if, instead, we tried to encode information into the two-dimensional phase space?

Suppose we want to simulate a solution with initial conditions $(P_{\text{ant}}, P_{\text{aphid}}) = (3, 21)$ at time 0. We would first compute $(P'_{\text{ant}}, P'_{\text{aphid}}) = (3.5, -8.1)$ and our estimate for $(P_{\text{ant}}, P_{\text{aphid}})$ at time Δ would be

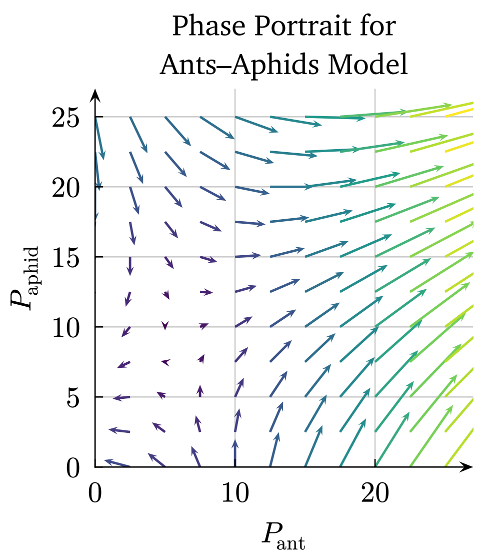
$$(P_{\text{ant}}, P_{\text{aphid}}) + \Delta \cdot (P'_{\text{ant}}, P'_{\text{aphid}}) \quad = \quad (3, 21) + \Delta \cdot (3.5, -8.1).$$

In other words, we start at $(3, 21)$ and move slightly in the direction of $(3.5, -8.1)$ (i.e., in the direction of $(P'_{\text{ant}}, P'_{\text{aphid}})$).

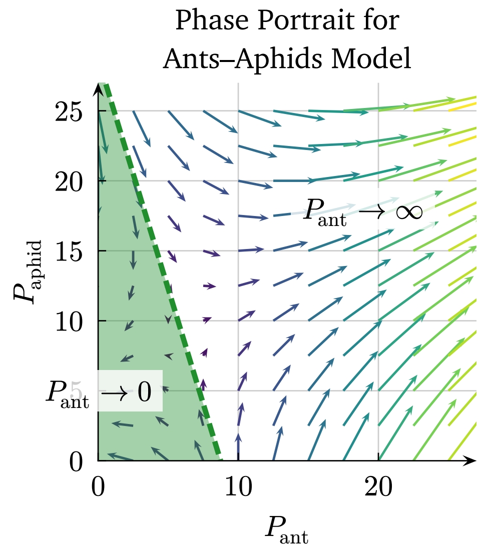


What if, we drew the vector $(P'_{\text{ant}}, P'_{\text{aphid}})$ at many points in phase space? Doing so, we would produce a *phase portrait*.

Phase Portrait. A ***phase portrait*** or ***phase diagram*** is the plot of a vector field in phase space where each vector rooted at (x, y) is tangent to a solution curve passing through (x, y) and its length is given by the speed of a solution passing through (x, y).



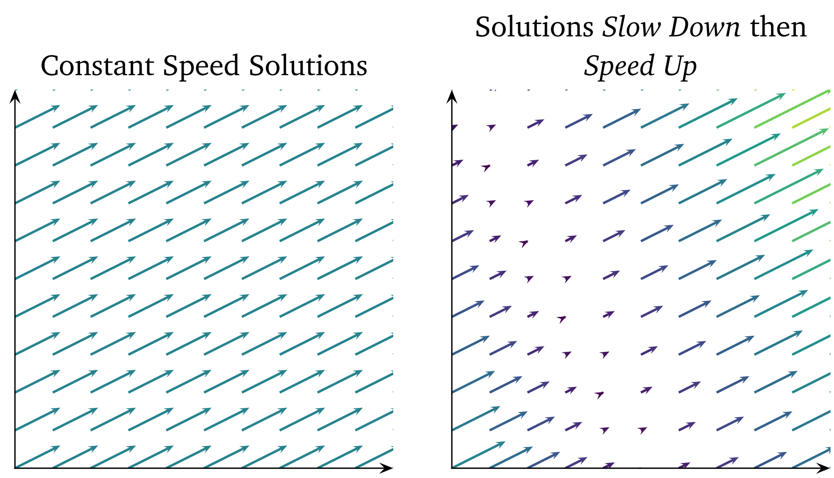
Analyzing the phase portrait, like the slope field, we can see solutions with different qualities. For example, we can divide phase space into sets of initial conditions where the ants population goes to zero and where the ants population is unbounded.



We could further subdivide phase space into regions where solutions (in phase space) are concave up or concave down, where ants/aphids achieve a maximum population, etc.. How we analyze a phase portrait depends on what questions we are trying to answer.

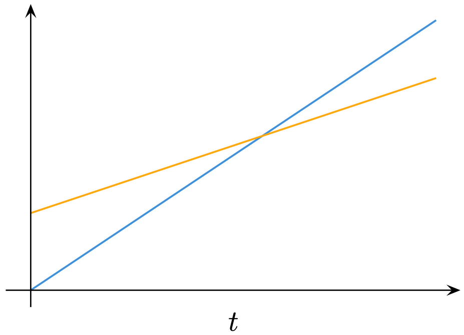
Phase Portraits and Missing Time

One important piece of information that is missing from phase portraits is a sense of time. While component space includes the independent variable, phase space does not. We partially make up for this by adjusting the length of an arrow in a phase portrait to correspond to the speed that a solution passes through the arrow at that point.

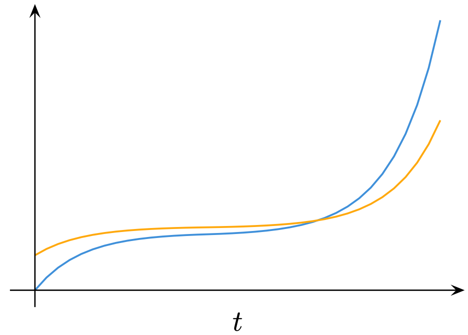


In the phase portraits above, solutions (in phase space) move in a straight line up and to the right. However, in the rightmost phase portrait, we can see that solutions slow down (almost to a stop) and then speed up again. When graphed in component space, the difference is visible.

Constant Speed Solutions
Component Space



Solutions *Slow Down* then *Speed Up*
Component Space



XXX Add example that asks, say, to draw a phase portrait for periodic solutions that are sines vs. that aren't.

An important note is that phase portraits are useful for *autonomous equations*, equations where the independent variable does not appear. Non-autonomous must be analyzed a different way.

Phase Portraits for Single Equations

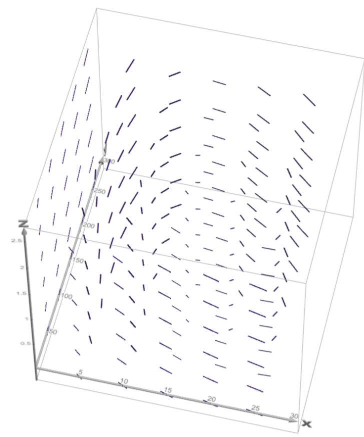
XXX Finish - write about 1d phase portraits

Practice Problems

1. Explain what you need to do in two different ways.
   (a) Way 1
   (b) Way 2
2. A second question.
3. A third question.

Solutions for Module 5

1. (a) Way 1 is good
   (b) Way 2 is better
2. The answer to the second question.
3.



https://www.desmos.com/3d/kvyztvmp0g

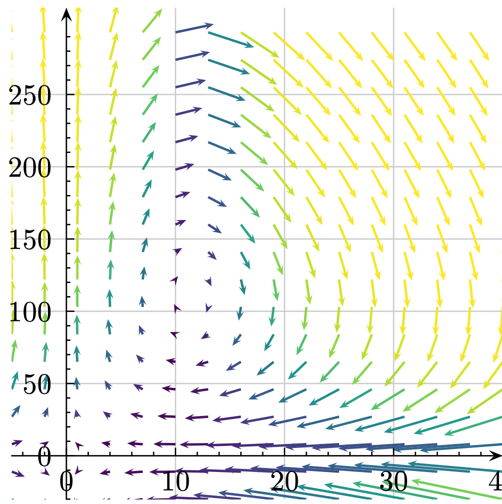
Three dimensional slope fields are possible, but hard to interpret. This is a slope field for the Foxes–Rabbits model.

24.1 What are the three dimensions in the plot?

24.2 What should the graph of an equilibrium solution look like?

24.3 What should the graph of a typical solution look like?

24.4 What are ways to simplify the picture so we can more easily analyze solutions?



Phase Portrait. A ***phase portrait*** or ***phase diagram*** is the plot of a vector field in phase space where each vector rooted at (x, y) is tangent to a solution curve passing through (x, y) and its length is given by the speed of a solution passing through (x, y).

This is a phase portrait for the Foxes–Rabbits model (introduced in Core Exercise 15).

`https://www.desmos.com/calculator/vrk0q4espx`

25.1 What do the x and y axes correspond to?

25.2 Identify the equilibria in the phase portrait. What are the lengths of the vectors at those points?

25.3 Classify each equilibrium as stable/unstable.

25.4 Copy and paste data from your simulation spreadsheet into the Desmos plot. Does the resulting curve fit with the picture?

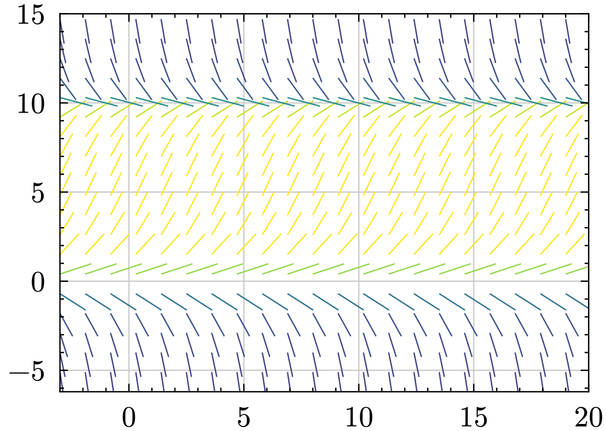
26 Sketch your own vector field where the corresponding system of differential equations:

26.1 Has an attracting equilibrium solution.

26.2 Has a repelling equilibrium solution.

26.3 Has no equilibrium solutions.

27



Recall the slope field for model **O**.

27.1 What would a phase portrait for model **O** look like? Draw it.

27.2 Where are the arrows the longest? Shortest?

27.3 How could you tell from a 1d phase portrait whether an equilibrium solution is attracting/repelling/etc.?

28 The following differential equation models the life cycle of a tree. In the model

- $H(t)$ = height (in meters) of tree trunk at time t
- $A(t)$ = surface area (in square meters) of all leaves at time t

$$H'(t) = 0.3 \cdot A(t) - b \cdot H(t)$$
$$A'(t) = -0.3 \cdot (H(t))^2 + A(t)$$

and $0 \leq b \leq 2$.

28.1 Modify

`https://www.desmos.com/calculator/vrk0q4espx`

to make a phase portrait for the tree model.

28.2 What do equilibrium solutions mean in terms of tree growth?

28.3 For $b = 1$ what are the equilibrium solution(s)?

29 29.1 Fix a value of b and use a spreadsheet to simulate some solutions with different initial conditions. Plot the results on your phase portrait from 28.1.

29.2 What will happen to a tree with $(H(0), A(0)) = (20, 10)$? Does this depend on b?

29.3 What will happen to a tree with $(H(0), A(0)) = (10, 10)$? Does this depend on b?

30 The tree model

$$H'(t) = 0.3 \cdot A(t) - b \cdot H(t)$$
$$A'(t) = -0.3 \cdot (H(t))^2 + A(t)$$

was based on the premises

$(\mathrm{P}_{\text{height 1}})$ CO_2 is absorbed by the leaves and turned directly into trunk height.
$(\mathrm{P}_{\text{height 2}})$ The tree is in a swamp and constantly sinks at a speed proportional to its height.
$(\mathrm{P}_{\text{leaves}})$ Leaves grow proportionality to the energy available.
$(\mathrm{P}_{\text{energy 1}})$ The tree gains energy from the sun proportionally to the leaf area.
$(\mathrm{P}_{\text{energy 2}})$ The tree loses energy proportionally to the square of its height.

30.1 How are the premises expressed in the differential equations?

30.2 What does the parameter b represent (in the real world)?

30.3 Applying Euler's method to this system shows solutions that pass from the 1st to 4th quadrants of the phase plane. Is this realistic? Describe the life cycle of such a tree?

31 Recall the tree model

$$H'(t) = 0.3 \cdot A(t) - b \cdot H(t)$$
$$A'(t) = -0.3 \cdot (H(t))^2 + A(t)$$

31.1 Find all equilibrium solutions for $0 \leq b \leq 2$.

31.2 For which b does a tree have the possibility of living forever? If the wind occasionally blew off a few random leaves, would that change your answer?

31.3 Find a value b_5 of b so that there is an equilibrium with $H = 5$.

Find a value b_{12} of b so that there is an equilibrium with $H = 12$.

31.4 Predict what happens to a tree near equilibrium in condition b_5 and a tree near equilibrium in condition b_{12}.

Linear Systems with Constant Coefficients (Real Solutions)

In this module you will learn:

- How to find explicit solutions to systems of differential equations with constant coefficients by using the eigenvalues and eigenvectors of the coefficient matrix.
- How the real, distinct eigenvalues of a system of differential equations with constant coefficients are related to the stability of the equilibrium solutions.

Linear Algebra Pre-requisites

Starting with this module, we will be using several results from linear algebra. We will assume that you are familiar with the following:

- Vector spaces and subspaces.
- Linear transformations and their null space/kernel and range/image.
- Linear independence of vectors.
- Span, basis, and dimension of a vector space/subspace.
- Eigenvalues and eigenvectors of a matrix and how to compute them.

You will find a review of these topics in Appendix F.

Finding Explicit Solutions

For equations like $y' = 7y$, we were lucky enough to guess-and-check our way to solutions $y(t) = Ae^{7t}$ where $A \in \mathbb{R}$ is a parameter. It turns out, by using some insights from linear algebra, we will be able to guess-and-check our way to solutions to some systems of differential equations as well.

Matrix Form

Consider the system

$$\begin{aligned} x' &= 2x + y \\ y' &= x + 2y \end{aligned}$$

We can rewrite this system as a matrix equation:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

We can further refine our matrix equation by introducing a function $\vec{r}(t) = \left[\begin{smallmatrix} x(t) \\ y(t) \end{smallmatrix}\right]$. Since the derivative of a multivariable function is the derivative of each of its components, $\vec{r}\,'(t) = \left[\begin{smallmatrix} x'(t) \\ y'(t) \end{smallmatrix}\right]$, and so the system can be rewritten as

$$\vec{r}\,' = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \vec{r}.$$

Differential equations written in this way are said to be expressed in *matrix form*.

Matrix Form. A system of differential equations

$$x_1'(t) = \cdots$$
$$x_2'(t) = \cdots$$
$$\vdots$$

is written in ***matrix form*** if it is expressed as

$$\vec{x}'(t) = M\vec{x}(t)$$

where $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \end{bmatrix}$ and M is a matrix with real entries.

Eigenvectors and Guessing Solutions

The equation $\vec{r}' = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\vec{r}$ looks a lot like our previous equation $y' = ky$, so we might get lucky and be able to guess a solution!

Let's start by guessing a solution of the form

$$\vec{r}(t) = \begin{bmatrix} A \\ B \end{bmatrix} e^{kt} = \begin{bmatrix} Ae^{kt} \\ Be^{kt} \end{bmatrix},$$

where $A, B, k \in \mathbb{R}$ are parameters. This implies

$$\vec{r}'(t) = \begin{bmatrix} kAe^{kt} \\ kBe^{kt} \end{bmatrix} = k\begin{bmatrix} Ae^{kt} \\ Be^{kt} \end{bmatrix} = k\,\vec{r}(t).$$

From the original differential equation, we know $\vec{r}' = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\vec{r}$. Combining this with our previous computation gives us

$$\vec{r}' = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\vec{r} = k\,\vec{r}.$$

In other words, $\vec{r}$ must be an eigenvector of $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ with eigenvalue k!

Computing, we see the eigenvectors of $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ are

$$\vec{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ with eigenvalue } 3 \qquad \text{and} \qquad \vec{v_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ with eigenvalue } 1.$$

Thus we guess solutions

$$\vec{r_1}(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} = \begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix} \qquad \text{and} \qquad \vec{r_2}(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{t} = \begin{bmatrix} e^{t} \\ -e^{t} \end{bmatrix}.$$

Verifying, we see that both $\vec{r_1}$ and $\vec{r_2}$ are solutions!

The Subspace of Solutions

Recall the system of differential equations

$$x' = 2x + y$$
$$y' = x + 2y$$

expressed in matrix form as

$$\vec{r}' = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\vec{r}$$

where $\vec{r}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$. By guessing and checking, we found two solutions

$$\vec{r}_1(t) = e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \vec{r}_2(t) = e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

but are there others?

In the case of the single-variable equation $y' = 7y$, we noticed that multiplying a solution by a constant gave another solution. Will the same work in this case? Let's try.

Example. Decide whether $\alpha\,\vec{r}_1$ and $\beta\,\vec{r}_2$ are solutions, where $\alpha, \beta \in \mathbb{R}$ are constants.

We can test if something is a solution to a differential equation by plugging it in. On the one hand, we have

$$(\alpha\,\vec{r}_1(t))' = \alpha\,\vec{r}_1'(t) = \alpha\,3\,\vec{r}_1(t).$$

On the other hand, we have

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}(\alpha\,\vec{r}_1(t)) = \alpha\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\vec{r}_1(t) = \alpha\,3\,\vec{r}_1(t),$$

and so $(\alpha\,\vec{r}_1(t)) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}(\alpha\,\vec{r}_1(t))$, showing that $\alpha\,\vec{r}_1(t)$ is a solution no matter the value of α. A similar computation shows that $\beta\,\vec{r}_2(t)$ is a solution for any value of β.

The preceding example can be generalized to show that if you have a solution to a matrix differential equation, then all scalar multiples of that solution are also solutions. This is in turn a special case of a more general theorem.

Theorem (Solutions Form a Subspace)

Let $\vec{r}' = M\vec{r}$ be a matrix differential equation and let $\mathcal{S}$ be the set of all solutions. Then, $\mathcal{S}$ is a subspace. In particular, $\mathcal{S}$ is closed under linear combinations.

Proof: Suppose that $\mathcal{S}$ is the set of all solutions to $\vec{r}' = M\vec{r}$ (where M is a square matrix). First, notice that

$$\vec{0}' = \vec{0} = M\vec{0},$$

so $\vec{0} \in \mathcal{S}$. Next, suppose that $\vec{s}_1$ and $\vec{s}_2$ are solutions in $\mathcal{S}$. By definition, that means $\vec{s}_1' = M\vec{s}_1$ and $\vec{s}_2' = M\vec{s}_2$. By the linearity of the derivative, we may now compute

$$\begin{aligned}(\alpha\,\vec{s}_1 + \beta\,\vec{s}_2)' &= \alpha\,\vec{s}_1' + \beta\,\vec{s}_2' \\ &= \alpha\,M\vec{s}_1 + \beta\,M\vec{s}_2 \\ &= M(\alpha\,\vec{s}_1 + \beta\,\vec{s}_2),\end{aligned}$$

and so $\alpha\,\vec{s}_1 + \beta\,\vec{s}_2$ is also a solution in $\mathcal{S}$.

■

Given the above theorem, we can find all sorts of solutions to $\vec{r}'(t) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\vec{r}(t)$, like

$$\vec{r}_1(t) + 4\,\vec{r}_2(t) = \begin{bmatrix} e^{3t} + 4\,e^{t} \\ e^{3t} - 4\,e^{t} \end{bmatrix} \qquad \text{or} \qquad -2\,\vec{r}_1(t) + 3\,\vec{r}_2(t) = \begin{bmatrix} -2\,e^{3t} + 3\,e^{t} \\ -2\,e^{3t} - 3\,e^{t} \end{bmatrix}.$$

But, have we found all the solutions? To answer that question we need to dive deeper into the linear algebra of the solution space.

Linear Algebra of the Solution Space

You're familiar with the vector spaces $\mathbb{R}^n$, but the set of all function from $\mathbb{R}$ to $\mathbb{R}^n$, denoted $\mathcal{F}^n$, also forms a vector space. To quickly check, notice that the constant function $z(t) = 0$ acts like the "zero vector", and if $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are functions, then so is $\alpha \cdot f + \beta \cdot g$.[14]

The space $\mathcal{F}^n$ is *large*. In fact, it is *infinite dimensional*, but the theorems of linear algebra still apply (provided the proper definitions).

Linear Dependence & Independence of Functions. The functions $\vec{v}_1(t), \vec{v}_2(t), \ldots, \vec{v}_n(t)$ are ***linearly dependent*** if there is a non-trivial linear combination of $\vec{v}_1(t), \ldots, \vec{v}_n(t)$ that equals the zero for all $t \in \mathbb{R}$.

Otherwise they are ***linearly independent***.

Example. Let $f(x) = x^2$, $g(x) = x$, and $h(x) = 2x^2 - x$. Show that $\{f, g\} \subseteq \mathcal{F}^1$ is linearly independent but $\{f, g, h\} \subseteq \mathcal{F}^1$ is linearly dependent

Consider the set $\{f, g\}$. We can attempt to solve the equation

$$\alpha f(x) + \beta g(x) = 0$$

Expanding, we see

$$\alpha f(x) + \beta g(x) = \alpha x^2 + \beta x = 0.$$

Since this equation must hold true for all x, it must hold true for $x = 1$. Therefore,

$$\alpha f(1) + \beta g(1) = \alpha 1^2 + \beta 1 = \alpha + \beta = 0.$$

Similarly, it must hold true for $x = 2$, and so

$$\alpha f(2) + \beta g(2) = \alpha 2^2 + \beta 2 = 4\alpha + 2\beta = 0.$$

From the first equation, we see that $\alpha = -\beta$ and from the second that $\alpha = -\frac{1}{2}\beta$. The only way for both of these conditions to be satisfied is if $\alpha = 0$ and $\beta = 0$. Thus $\{f, g\}$ is linearly independent.

Alternatively, consider the set $\{f, g, h\}$. We can attempt to solve the equation

$$\alpha f(x) + \beta g(x) + \gamma h(x) = \alpha x^2 + \beta x + \gamma(2x^2 - x) = 0,$$

which, by inspection, has a non-trivial solution $\alpha = -2$, $\beta = 1$, and $\gamma = 1$. Thus $\{f, g, h\}$ are linearly dependent.

A historical technique for determining whether a set of functions is linearly independent is to compute the determinant of the *Wronskian matrix*;[15] while appeal to the Wronskian is occasionally useful, we will stick with direct approaches in this text.

With the definition of linear independence/dependence pinned down, we can apply the usual Linear Algebra definitions of subspace, basis, and dimension to $\mathcal{F}^n$.

[14] To fully check, you must check that all the *vector space* axioms hold.

[15] See `https://en.wikipedia.org/wiki/Wronskian`

Dimension of the solution space

So far, we have established that solutions to matrix differential equations form a subspace, and as a subspace space, they must have a dimension[16]. Our next goal will be to establish the dimension of the solution space to a matrix differential equation and then to find a general solution.

The following theorem will give us a place to start.

> **Theorem (Existence & Uniqueness 1)**
>
> The system of differential equations represented by $\vec{r}'(t) = M\vec{r}(t) + \vec{p}$ (or the single differential equation $y' = ay + b$) has a unique solution passing through every initial condition. Further, the domain of every solution is $\mathbb{R}$.

Now, suppose that M is an $n \times n$ matrix. Let $\mathcal{S}$ be the set of all solutions to $\vec{r}' = M\vec{r}$. The above theorem states that every initial value problem

$$\vec{r}' = M\vec{r} \qquad \text{and} \qquad \vec{r}(t_0) = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

has a unique solution. Thus, the space of all solutions can be no “bigger” than the set of all initial conditions. Notice that we have $n+1$ parameters to choose for initial conditions. We have n choices coming from x_1, …, x_n with one more choice coming from the value of t_0.

This means that the dimension of $\mathcal{S}$ is bounded above by $n+1$.[17] But, we can do slightly better.

> **Theorem (Solutions pass through $t = 0$)**
>
> Let $\vec{w}$ be a solution to the initial value problem $\vec{r}' = M\vec{r}$ with initial condition $\vec{r}(t_0) = \vec{r}_0$. Then, $\vec{w}$ is ***also*** the solution to the initial value problem $\vec{r}' = M\vec{r}$ with initial condition $\vec{r}(0) = \vec{w}_0$ where $\vec{w}_0 = \vec{w}(0)$.

Proof: Let $\vec{w}$ be a solution to the initial value problem $\vec{r}' = M\vec{r}$ with initial condition $\vec{r}(t_0) = \vec{r}_0$. Based on the Existence & Uniqueness Theorem 1, we know $\vec{w}_0 = \vec{w}(0)$ is defined. Further, since there is a solution to every initial value problem, $\vec{r}' = M\vec{r}$ with $\vec{r}(0) = \vec{w}_0$ has a solution. Finally, because every solution passing through a set of initial conditions is unique, since $\vec{w}(t)$ already passes through $\vec{w}_0$, we must have that $\vec{w}$ is the unique solution to the initial value problem $\vec{r}' = M\vec{r}$ with initial condition $\vec{r}(0) = \vec{w}_0$. ■

The preceding theorem shrinks the space of solutions. Originally we knew that the set of all solutions can be obtained by solving initial value problems for all choices of x_1, …, x_n, **and** t_0. But now we know that fixing $t_0 = 0$ gives us the same solutions. Thus, by making n choices, we can determine a solution. This means

$$\dim(\mathcal{S}) \leq n.$$

This argument leads to the following theorem.

---

[16] This is a powerful theorem coming from abstract Linear Algebra and relying on the axiom of choice. We will just accept this theorem as fact.

[17] To make this argument fully rigorous, you need to establish the existence of a smooth map between the space of initial conditions and the space of solutions and then invoke the theorem that smooth maps cannot increase the dimension of a space.

Theorem (Solution Space Upper Bound)

Let M be an $n \times n$ matrix and let $\mathcal{S}$ be the set of all solutions to $\vec{r}'(t) = M\vec{r}(t)$. Then

$$\dim(\mathcal{S}) \leq n.$$

Finding a basis of solutions

Recall that M is an $n \times n$ matrix and that $\mathcal{S}$ is the set of all solutions to $\vec{r}' = M\vec{r}$. We have established $\dim(\mathcal{S}) \leq n$. We will now try to find the exact dimension and a basis for $\mathcal{S}$.

Before reading this part of the module, you should work through the core exercises 32 – 38 to understand where the idea comes from. XXX Think about whether this becomes a consistent margin note, at the beginning of the module, or something different.

Suppose that $\vec{v}$ is an eigenvector for M with eigenvalue λ and consider $\vec{s}(t) = \vec{v}e^{\lambda t}$. Multiplying by M we get

$$M\vec{s}(t) = M\big(\vec{v}e^{\lambda t}\big) = M\vec{v}e^{\lambda t} = \lambda\vec{v}e^{\lambda t} = \lambda\vec{s}(t).$$

Similarly, taking a derivative with respect to t, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{s}(t) = \lambda\vec{v}e^{\lambda t} = \lambda\vec{s}(t).$$

In other words, $\vec{s}' = \lambda\vec{s} = M\vec{s}$, and so $\vec{s}$ is a solution to our differential equation. This means that whenever we have an eigenvector/eigenvalue for M, we can write down an explicit solution to $\vec{r}' = M\vec{r}$. Solutions constructed this way are called *eigen solutions*.

Eigen Solution. Let M be an $n \times n$ matrix and let $\vec{v}$ be an eigenvector for M with associated eigenvalue λ. The function

$$\vec{r}(t) = \vec{v}e^{\lambda t}$$

is called an ***eigen solution*** of the differential equation $\vec{r}' = M\vec{r}$.

We just proved that any eigen solution to $\vec{r}' = M\vec{r}$ is, indeed, a solution. Since eigen solutions corresponding to linearly independent eigenvectors are linearly independent (justify this fact to yourself), we know

$$\dim(\mathcal{S}) \geq \#\ \text{linearly independent eigenvectors of } M.$$

Thus, if M has n linearly independent eigenvectors (i.e., if M is diagnoalizable), then

$$\dim(\mathcal{S}) \geq n.$$

Combined with our previous result, we now have the following theorem.

Theorem (Solution Space Dimension)

Let M be an $n \times n$ matrix and let $\mathcal{S}$ be the set of all solutions to $\vec{r}'(t) = M\vec{r}(t)$. Then, if M is diagonalizable,

$$\dim(\mathcal{S}) = n.$$

In this text, we will only consider matrix differential equations where the matrix is diagonalizable, but the above theorem also holds when M is *not* diagonalizable. However, the proof relies on finding a basis of solutions, some of which are not eigen solutions.

Solutions of systems with real eigenvalues

The preceding section shows us that if we have a full set of eigenvectors[18] for a matrix M, *every solution* can be expressed as a linear combination of the eigen solutions.

Example. Find all solutions of the system of differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 2x + y$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = x + 2y$$

We can write this system in matrix form as:

$$\begin{bmatrix} \frac{\mathrm{d}x}{\mathrm{d}t} \\ \frac{\mathrm{d}y}{\mathrm{d}t} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

and if we let $\vec{r} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $M = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, we can write the system as:

$$\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = M\vec{r},$$

where $\vec{r}$ is a vector of the two dependent variables and M is the coefficient matrix.

We can find the eigenvalues and eigenvectors of the matrix M:

- $\lambda_1 = 3$ with eigenvector $\vec{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$,
- $\lambda_2 = 1$ with eigenvector $\vec{v_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

We can now write the eigen solutions of the system of differential equations:

$$\vec{r_1}(t) = \vec{v_1}e^{3t} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t},$$
$$\vec{r_2}(t) = \vec{v_2}e^{1t} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{1t}.$$

Linear combination of these two solutions form general solution:

$$\vec{r}(t) = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + \beta \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{1t},$$

where α and β are constants that depend on the initial conditions of the system.

Example. Find the solution to $\vec{r}' = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\vec{r}$ that satisfies $\vec{r}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

We already found the general solution to $\vec{r}' = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\vec{r}$:

$$\vec{r}(t) = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + \beta \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{1t},$$

where α and β are parameters.

Since we require $\vec{r}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, we can solve

[18] "full set" means that for an $n \times n$ matrix we have n linearly independent eigenvectors.

$$\vec{r}(0) = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3\cdot 0} + \beta \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{1\cdot 0}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

to get $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{2}$. Thus the solution that satisfies the initial condition is:

$$\vec{r}(t) = \left(\frac{1}{2}\right) \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + \left(\frac{1}{2}\right) \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{1t}$$

$$= \left(\frac{1}{2}\right) \begin{bmatrix} e^{3t} + e^{t} \\ e^{3t} - e^{t} \end{bmatrix}.$$

Eigen solutions in phase space

XXX Finish

Stability of the Equilibrium Solutions

Now that we have a method to find the solutions of a system of differential equations with constant coefficients, we can study the question:

> As time goes to infinity, what is the long-term behaviour of solutions? Do they tend towards or away from the equilibrium solution(s)?

First note that if $\vec{s}$ is an equilibrium solution to $\vec{r}' = M\vec{r}$, then $\vec{s}'(t) = \vec{0}$ which implies $M\vec{s} = \vec{0}$. This means that if M is invertible, the equation $\vec{r}' = M\vec{r}$ has a unique equilibrium solution: $\vec{s}'(t) = \vec{0}$. Further, all equilibrium solutions can be found by analyzing the null space of M.

For illustration purposes, let us now consider a 2×2 matrix M with real and distinct eigenvalues $\lambda_1 \neq \lambda_2$ and corresponding eigenvectors $\vec{v}_1$ and $\vec{v}_2$. The general solution to the equation $\vec{r}' = M\vec{r}$ is

$$\vec{r}(t) = c_1\vec{v}_1 e^{\lambda_1 t} + c_2\vec{v}_2 e^{\lambda_2 t},$$

where c_1 and c_2 parameters. We want to study the long-term behaviour of solutions, which depends on the long-term behaviour of the exponential function:

$$\lim_{t\to\infty} e^{\lambda t} = \begin{cases} 0 & \text{if } \ \lambda < 0 \\ 1 & \text{if } \ \lambda = 0. \\ \infty & \text{if } \ \lambda > 0 \end{cases}$$

We immediately see:

- If $\lambda_1 < 0$ and $\lambda_2 < 0$, then $\lim_{t\to\infty} \vec{r}(t) = \vec{0}$ no matter the values of c_1 and c_2.
- If $\lambda_1 > 0$ and $\lambda_2 > 0$, then $\lim_{t\to\infty} \vec{r}(t)$ diverges as long as at least one of c_1 and c_2 are non-zero.

What can this tell us about the stability of the equilibrium solution?

Recall the informal classification of equilibrium solutions[19] from Module 4.

[19] Check Module 4 for the precise definition.

Classification of Equilibria. An equilibrium solution f is called

- *attracting* if locally, solutions converge to f;
- *repelling* if there is a fixed distance so that locally, solutions tend away from f by that fixed distance;
- *stable* if for any fixed distance, locally, solutions stay within that fixed distance of f; and,
- *unstable* if f is not stable.

Now consider the equilibrium solution $\vec{s}(t) = \vec{0}$.[20] If $\vec{r}(t) = c_1\vec{v}_1 e^{\lambda_1 t} + c_2\vec{v}_2 e^{\lambda_2 t}$ is "close to" $\vec{s}(t) = \vec{0}$ at time $t = 0$, then c_1 and c_2 are close to zero. We can now consider cases based on λ_1 and λ_2:

- If λ_1 or λ_2 is positive, then there will be solutions that diverge $\implies \vec{s}$ is unstable.
- If λ_1 and λ_2 are negative, then *all* solutions will converge zero $\implies \vec{s}$ is stable and attracting.
- If one of λ_1 or λ_2 is zero, then there are constant solutions near $\vec{s} \implies \vec{s}$ is not attracting nor repelling.
  - If $\lambda_1 = 0$ and $\lambda_2 < 0$, then $\vec{s}$ will be stable, because solutions either converge to the equilibrium ($\vec{0}$) or are constant solutions.
  - If $\lambda_1 = 0$ and $\lambda_2 > 0$, then $\vec{s}$ will be unstable, because solutions either diverge to infinity or are constant solutions.

The following tables summarize our findings:

Eigenvalue λ_1	Eigenvalue λ_2	Stability of Equilibrium
$\lambda_1 < 0$	$\lambda_2 < 0$	Stable and Attracting
$\lambda_1 > 0$	$\lambda_2 > 0$	Unstable and Repelling
$\lambda_1 < 0$	$\lambda_2 > 0$	Unstable
$\lambda_1 = 0$	$\lambda_2 = 0$	Stable
$\lambda_1 = 0$	$\lambda_2 < 0$	Stable
$\lambda_1 = 0$	$\lambda_2 > 0$	Unstable

or

Sign of Eigenvalues	Stability of Equilibrium
Both negative	Stable and Attracting
Both positive	Unstable and Repelling
One negative, one positive	Unstable
Both zero	Stable
One is zero, one is negative	Stable
One is zero, one is positive	Unstable

Example. Suppose $\vec{r}' = M\vec{r}$ has a general solution

$$\vec{r}(t) = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t} + \beta \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-t}.$$

Prove that the equilibrium solution $\vec{s}(t) = \vec{0}$ unstable and ***not*** repelling.

XXX Finish

[20] There may be other equilibrium solutions, but this equilibrium solution always exists.

Practice Problems

1. Let S be the set of all solutions to

$$\vec{r}(t) = \begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}\vec{r}(t).$$

   (a) Show that $\dim(S) \geq 2$ by finding at least two linearly independent solutions.

   (b) Let I be the set of all initial conditions. What is I?

   (c) Show that $\dim(S) \leq 3$ by applying the theorem to the set of initial conditions.

   (d) Can two points in I correspond to the same solution? Explain?

   (e) Find a subset $U \subseteq I$ so that every solution corresponds to a unique point in U.

   (f) Show that $\dim(S) \leq 2$.

   (g) Suppose M is an $n \times n$ matrix. Consider the differential equation $\vec{r}'(t) = M\vec{r}(t)$. If you have found n linearly independent solutions, can you determine the dimension of the set of all solutions? Explain.

2. A second question.
3. A third question.

Solutions for Module 6

1. (a) Way 1 is good
   (b) Way 2 is better
2. The answer to the second question.
3.

32 Consider the system of differential equations

$$x'(t) = x(t)$$
$$y'(t) = 2y(t)$$

32.1 Make a phase portrait for the system.

https://www.desmos.com/calculator/h3wtwjghv0

32.1 What are the equilibrium solution(s) of the system?

32.2 Find a formula for $x(t)$ and $y(t)$ that satisfy the initial conditions $(x(0), y(0)) = (x_0, y_0)$.

32.3 Let $\vec{r}(t) = (x(t), y(t))$. Find a matrix A so that the differential equation can be equivalently expressed as

$$\vec{r}'(t) = A\vec{r}(t).$$

32.4 Write a solution to $\vec{r}' = A\vec{r}$ (where A is the matrix you came up with).

33 Let A be an unknown matrix and suppose $\vec{p}$ and $\vec{q}$ are solutions to $\vec{r}' = A\vec{r}$.

33.1 Is $\vec{s}(t) = \vec{p}(t) + \vec{q}(t)$ a solution to $\vec{r}' = A\vec{r}$? Justify your answer.

33.2 Can you construct other solutions from $\vec{p}$ and $\vec{q}$? If yes, how so?

Linear Dependence & Independence (Algebraic). The vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are ***linearly dependent*** if there is a non-trivial linear combination of $\vec{v}_1, \ldots, \vec{v}_n$ that equals the zero vector. Otherwise they are ***linearly independent***.

Define

$$\vec{p}(t) = \begin{pmatrix} e^t \\ 0 \end{pmatrix} \qquad \vec{q}(t) = \begin{pmatrix} 4e^t \\ 0 \end{pmatrix} \qquad \vec{h}(t) = \begin{pmatrix} 0 \\ e^{2t} \end{pmatrix} \qquad \vec{z}(t) = \begin{pmatrix} 0 \\ e^{3t} \end{pmatrix}.$$

34.1 Are $\vec{p}$ and $\vec{q}$ linearly independent or linearly dependent? Justify with the definition.

34.2 Are $\vec{p}$ and $\vec{h}$ linearly independent or linearly dependent? Justify with the definition.

34.3 Are $\vec{h}$ and $\vec{z}$ linearly independent or linearly dependent? Justify with the definition.

34.4 Is the set of three functions $\{\vec{p}, \vec{h}, \vec{z}\}$ linearly independent or linearly dependent? Justify with the definition.

35 Recall

$$\vec{p}(t) = \begin{bmatrix} e^t \\ 0 \end{bmatrix} \qquad \vec{q}(t) = \begin{bmatrix} 4e^t \\ 0 \end{bmatrix} \qquad \vec{h}(t) = \begin{bmatrix} 0 \\ e^{2t} \end{bmatrix} \qquad \vec{z}(t) = \begin{bmatrix} 0 \\ e^{3t} \end{bmatrix}.$$

35.1 Describe $\operatorname{span}\left\{\vec{p}, \vec{h}\right\}$. What is its dimension? What is a basis for it?

35.2 Let S be the set of all solutions to $\vec{r}'(t) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \vec{r}(t)$. (You've seen this equation before.)

Is S a subspace? If so, what is its dimension?

35.3 Provided S is a subspace, give a basis for S.

36 Consider the differential equation

$$y'(t) = 2 \cdot y(t).$$

36.1 Write a solution whose graph passes through the point $(t, y) = (0, 3)$.

36.2 Write a solution whose graph passes through the point $(t, y) = (0, y_0)$.

36.3 Write a solution whose graph passes through the point $(t, y) = (t_0, y_0)$.

36.4 Consider the following argument:

> For every point (t_0, y_0), there is a corresponding solution to $y'(t) = 2 \cdot y(t)$.
>
> Since $\{(t_0, y_0) : t_0, y_0 \in \mathbb{R}\}$ is two dimensional, this means the set of solutions to $y'(t) = 2 \cdot y(t)$ is two dimensional.

Do you agree? Explain.

Theorem (Existence & Uniqueness 1)

The system of differential equations represented by $\vec{r}\,'(t) = M\vec{r}(t) + \vec{p}$ (or the single differential equation $y' = ay + b$) has a unique solution passing through every initial condition. Further, the domain of every solution is $\mathbb{R}$.

Let S be the set of all solutions to $\vec{r}(t) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\vec{r}(t)$.

37.1 Show that $\dim(S) \geq 2$ by finding at least two linearly independent solutions.

37.2 Let I be the set of all initial conditions. What is I?

37.3 Show that $\dim(S) \leq 3$ by applying the theorem to the set of initial conditions.

37.4 Can two points in I correspond to the same solution? Explain?

37.5 Find a subset $U \subseteq I$ so that every solution corresponds to a unique point in U.

37.6 Show that $\dim(S) \leq 2$.

37.7 Suppose M is an $n \times n$ matrix. Consider the differential equation $\vec{r}\,'(t) = M\vec{r}(t)$. If you have found n linearly independent solutions, can you determine the dimension of the set of all solutions? Explain.

38 Consider the system

$$x'(t) = 2x(t)$$
$$y'(t) = 3y(t)$$

38.1 Rewrite the system in matrix form.

38.2 Classify the following as solutions or non-solutions to the system.

$$\vec{r}_1(t) = e^{2t} \qquad \vec{r}_2(t) = \begin{bmatrix} e^{2t} \\ 0 \end{bmatrix}$$

$$\vec{r}_3(t) = \begin{bmatrix} e^{2t} \\ 4e^{3t} \end{bmatrix} \qquad \vec{r}_4(t) = \begin{bmatrix} e^{3t} \\ e^{2t} \end{bmatrix}$$

$$\vec{r}_5(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

38.1 State the definition of an eigenvector for the matrix M.

38.2 What should the definition of an *eigen solution* be for this system?

38.3 Which functions from 38.2 are eigen solutions?

38.4 Find an eigen solution $\vec{r}_6$ that is linearly independent from $\vec{r}_2$.

38.5 Let $S = \mathrm{span}\{\vec{r}_2, \vec{r}_6\}$. Does S contain *all* solutions to the system? Justify your answer.

39 Recall the system

$$x'(t) = 2x(t)$$
$$y'(t) = 3y(t)$$

has eigen solutions $\vec{r}_2(t) = \begin{bmatrix} e^{2t} \\ 0 \end{bmatrix}$ and $\vec{r}_6(t) = \begin{bmatrix} 0 \\ e^{3t} \end{bmatrix}$

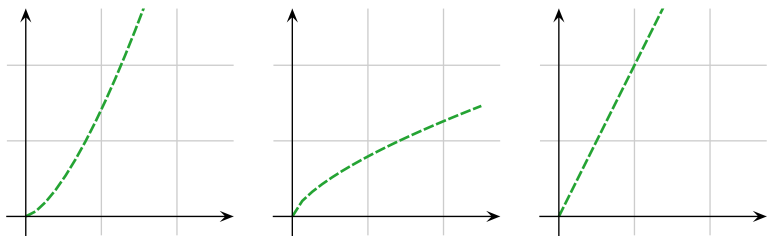
39.1 Sketch $\vec{r}_2$ and $\vec{r}_6$ in the phase plane.

39.2 Use

`https://www.desmos.com/calculator/h3wtwjghv0`

to make a phase portrait for the system.

39.3



In which phase plane above is the dashed (green) curve the graph of a solution to the system? Explain.

40 Suppose A is a 2×2 matrix and $\vec{s}_1$ and $\vec{s}_2$ are eigen solutions to $\vec{r}' = A\vec{r}$ with eigenvalues 1 and -1, respectively.

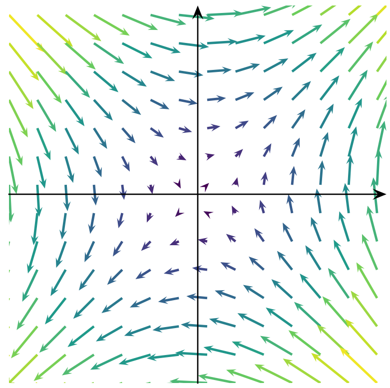
40.1 Write possible formulas for $\vec{s}_1(t)$ and $\vec{s}_2(t)$.

40.2 Sketch a phase plane with graphs of $\vec{s}_1$ and $\vec{s}_2$ on it.

40.3 Add a non-eigen solution to your sketch.

40.4 Sketch a possible phase portrait for $\vec{r}' = A\vec{r}$. Can you extend your phase portrait to all quadrants?

41 Consider the following phase portrait for a system of the form $\vec{r}' = A\vec{r}$ for an unknown matrix A.



41.1 Can you identify any eigen solutions?

41.2 What are the eigenvalues of A? What are their signs?

42 Consider the differential equation $\vec{r}'(t) = M\vec{r}(t)$ where $M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

42.1 Verify that $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ are eigenvectors for M. What are the corresponding eigenvalues?

42.2 (a) Is $\vec{r}_1(t) = e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ a solution to the differential equation? An eigen solution?

(b) Is $\vec{r}_2(t) = e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ a solution to the differential equation? An eigen solution?

(c) Is $\vec{r}_3(t) = e^{2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ a solution to the differential equation? An eigen solution?

42.3 Find an eigen solution for the system corresponding to the eigenvalue -1. Write your answer in vector form.

42.4 Let $\vec{v}$ be an eigenvector for M with eigenvalue λ. Explain how to write down an eigen solution to $\vec{r}'(t) = M\vec{r}(t)$ with eigenvalue λ.

42.5 Let $\vec{v} \neq \vec{0}$ be a non-eigenvector for M. Could $\vec{r}(t) = e^{\lambda t}\vec{v}$ be a solution to $\vec{r}'(t) = M\vec{r}(t)$ for some λ? Explain.

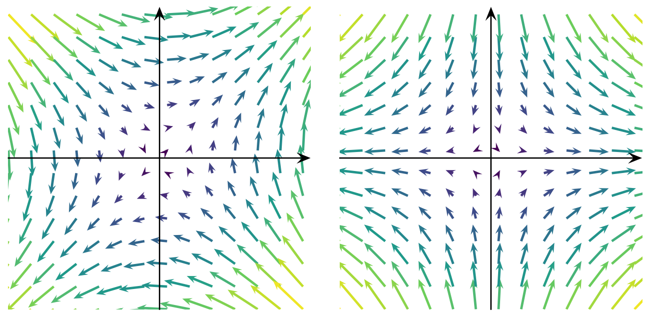
43 Recall the differential equation $\vec{r}'(t) = M\vec{r}(t)$ where $M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

43.1 Write down a general solution to the differential equation.

43.2 Write down a solution to the initial value problem $\vec{r}(0) = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$.

43.3 Are your answers to the first two parts the same? Do they contain the same information?

44 The phase portrait for a differential equation arising from the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (left) and $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ (right) are shown.

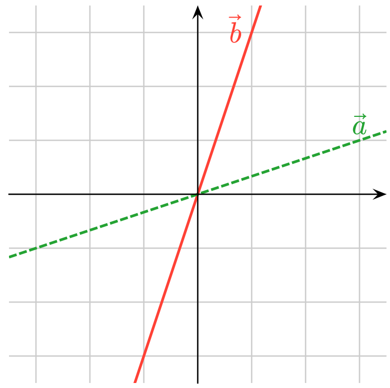


Both have eigenvalues ± 1, but they have different eigenvectors.

44.1 How are the phase portraits related to each other?

44.2 Suppose P is a 2×2 matrix with eigenvalues ± 1. In what ways could the phase portrait for $\vec{r}\,'(t) = P\vec{r}(t)$ look *different* from the above portraits? In what way(s) must it look the same?

45 Consider the following phase plane with lines in the direction of $\vec{a}$ (dashed green) and $\vec{b}$ (red).



45.1 Sketch a phase portrait where the directions $\vec{a}$ and $\vec{b}$ correspond to eigen solutions with eigenvalues that are:

	sign for $\vec{a}$	sign for $\vec{b}$
1	pos	pos
2	neg	neg
3	neg	pos
4	pos	neg
5	pos	zero

45.2 Classify the solution at the origin for situations (1)–(5) as stable or unstable.

45.3 Would any of your classifications in the previous part change if the directions of $\vec{a}$ and $\vec{b}$ changed?

46 You are examining a differential equation $\vec{r}'(t) = M\vec{r}(t)$ for an unknown 2×2 matrix M.

You would like to determine whether $\vec{r}(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is stable, unstable, attracting, or repelling.

46.1 Come up with a rule to determine the nature of the equilibrium solution $\vec{r}(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ based on the eigenvalues of M (provided there exist two linearly independent eigen solutions).

46.2 Consider the system of differential equations

$$x'(t) = x(t) + 2 \cdot y(t)$$
$$y'(t) = 3 \cdot x(t) - 4 \cdot y(t)$$

(a) Classify the stability of the equilibrium solution $(x(t), y(t)) = (0, 0)$ using any method you want.

(b) Justify your answer analytically using eigenvalues.

Linear Systems with Constant Coefficients (Affine Solutions)

In this module you will learn

- What makes a differential equation *affine*.
- How to relate affine equations and matrix equations.
- How to solve differential equations of the form $\vec{r}\,' = M\vec{r} + \vec{b}$.

We have a firm grasp on how to solve matrix differential equations of the form $\vec{r}\,' = M\vec{r}$. What about equations of the form $\vec{r}\,' = M\vec{r} + \vec{b}$? Unfortunately, that extra "$+\vec{b}$" changes a lot. For example, the solution set is no longer a subspace!

> **Example.** Show that solutions to $y' = 2y + 1$ do ***not*** form a subspace.
>
> Since $y' = 2y + 1$ is separable, we can explicitly solve to find solutions:
>
> $$y(t) = Ae^{2t} - \frac{1}{2} \qquad \text{where } A \text{ is a parameter.}$$
>
> Picking the solution $y_1(t) = -\frac{1}{2}$ (where A corresponds to 0) and $y_2(t) = e^{2t} - \frac{1}{2}$ (where A corresponds to 1), let
>
> $$y_3(t) = y_1(t) + y_2(t) = -\frac{1}{2} + e^{2t} - \frac{1}{2} = e^{2t} - 1.$$
>
> Differentiating, $y_{3'}(t) = 2e^{2t} = 2y_3(t) + 2$, and so y_3 is ***not*** a solution. Therefore, the solutions of $y' = 2y + 1$ ***cannot*** form a subspace.

The issue is that $\vec{r}\,' = M\vec{r} + \vec{b}$ is no loner a matrix equation, it is an *affine equation*. Affine equations are matrix equations plus a vector of constants, and we will soon see how to compensate for that extra "$+\vec{b}$".

Centering about the Equilibrium

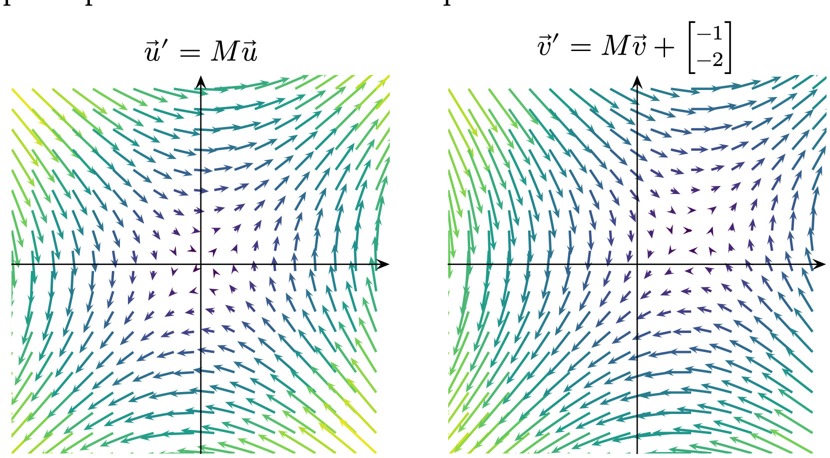
Let $M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and consider the equations

$$\vec{u}\,' = M\vec{u} \tag{1}$$

and

$$\vec{v}\,' = M\vec{v} + \begin{bmatrix} -1 \\ -2 \end{bmatrix}. \tag{2}$$

We can make phase portraits for both of these equations.



The phase portraits look very similar: they are shifts of each other. To figure out the exact shift, let's analyze the equilibrium points of both equations.

$$\vec{u}' = M\vec{u} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff \vec{u} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and

$$\vec{v}' = M\vec{v} + \begin{bmatrix} -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff \vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

and so the phase portraits are shifted by the vector $\left[\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right]$.

We can use this information to relate the solutions to $\vec{u}' = M\vec{u}$ and $\vec{v}' = M\vec{v} + \left[\begin{smallmatrix} -1 \\ -2 \end{smallmatrix}\right]$ to each other. Let $\vec{v}$ be a solution to Equation (2) and define the function

$$\vec{s}(t) = \vec{v}(t) - \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

We can now compute

$$\vec{s}'(t) = \left(\vec{v}(t) - \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)' = \vec{v}'(t) - \begin{bmatrix} 2 \\ 1 \end{bmatrix}' = \vec{v}'(t),$$

because the derivative of a constant vector is $\vec{0}$. Alternatively, multiplying by M we get

$$\begin{aligned} M\vec{s}(t) &= M\left(\vec{v}(t) - \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) \\ &= M\vec{v}(t) - M\begin{bmatrix} 2 \\ 1 \end{bmatrix} \\ &= M\vec{v}(t) - \begin{bmatrix} 1 \\ 2 \end{bmatrix} = M\vec{v}(t) + \begin{bmatrix} -1 \\ -2 \end{bmatrix} = \vec{v}'(t). \end{aligned}$$

In other words, $\vec{s}' = M\vec{s}$, and so $\vec{s}$ is a solution to Equation (1). What we have done is called a *change of variables* and the result is that we can relate solutions to the affine equation $\vec{v}' = M\vec{v} + \left[\begin{smallmatrix} -1 \\ -2 \end{smallmatrix}\right]$ with solutions to the matrix equation $\vec{u}' = M\vec{u}$.

Example. Let $M = \left[\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}\right]$ and suppose $\vec{v}$ is a solution to $\vec{v}' = M\vec{v} + \left[\begin{smallmatrix} 2 \\ 0 \end{smallmatrix}\right]$. Find a solution to $\vec{r}' = M\vec{r}$ in terms of $\vec{v}$.

XXX Finish

The processes we just followed (shifting a solution to the affine equation by its equilibrium point to find a solution to the corresponding matrix equation) always works, and we will refer to it as *centering about the equilibrium*.

Theorem (Centering about the Equilibrium)

Let M be a matrix and $\vec{b}$ a vector and consider the differential equations:

$$\vec{r}' = M\vec{r} \tag{3}$$

$$\vec{r}' = M\vec{r} + \vec{b} \tag{4}$$

If $\vec{p}$ is the equilibrium point of the affine equation (4) and $\vec{v}$ is a solution to the affine equation (4), then

$$\vec{s}(t) = \vec{v}(t) - \vec{p}$$

is a solution to the matrix equation (3).

Similarly, if $\vec{u}$ is a solution to the matrix equation (3), then

$$\vec{q}(t) = \vec{u}(t) + \vec{p}$$

is a solution to the affine equation (4).

Proof: We leave the proof of this Theorem as an exercise for the reader. XXX

Solving Affine Equations

Centring (and uncentering) about the equilibrium can be used to solve affine equations. Let's use our example from earlier:

$$\vec{r}' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \vec{r} + \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

By solving for $\vec{r}'(t) = \vec{0}$, we see that $\vec{r}(t) = \left[\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right]$ is an equilibrium solution. Next, we will apply our eigenvector techniques to solve the related equation $\vec{r}' = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right] \vec{r}$. This equation has solutions

$$\vec{u}(t) = Ae^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + Be^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \text{where } A \text{ and } B \text{ are parameters.}$$

These are not solutions to the original affine equation! However, we can define

$$\vec{s}(t) = \vec{u}(t) + \begin{bmatrix} 2 \\ 1 \end{bmatrix} = Ae^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + Be^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \text{where } A \text{ and } B \text{ are parameters.}$$

Now, we can check that $\vec{s}$ is a solution.

$$\vec{s}'(t) = \left(Ae^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + Be^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right)' = Ae^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} - Be^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

and

$$\begin{aligned} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \vec{s}(t) + \begin{bmatrix} -1 \\ -2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \left(Ae^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + Be^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right) + \begin{bmatrix} -1 \\ -2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \left(Ae^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + Be^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right) + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ -2 \end{bmatrix} \\ &= Ae^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} - Be^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ -2 \end{bmatrix} \\ &= Ae^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} - Be^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \end{aligned}$$

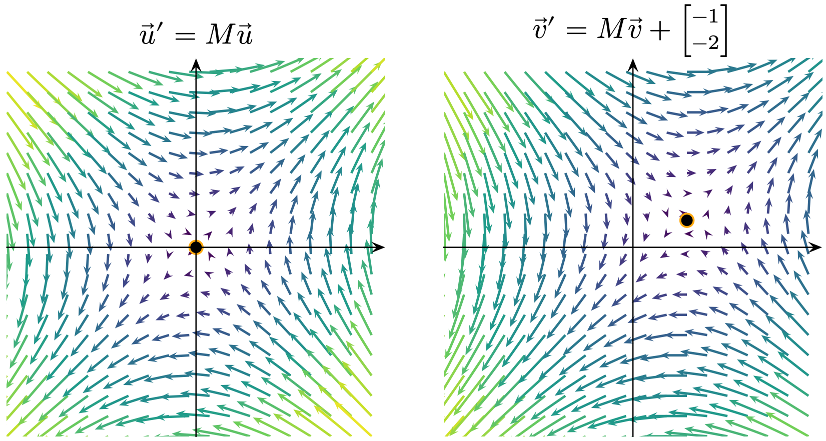
and so $\vec{s}$ is a solution for every choice of A and B.

Figuring out what to shift by

Do not try to *memorize* how to shift one solution to get another. Instead, spend some time visualizing the relationship between solutions to an affine equation and its corresponding matrix equation.

Remember: a matrix equation always has $\vec{r}(t) = \vec{0}$ as an equilibrium solution. So, if you think of the phase space picture from before and translate the equilibrium solution, all other solutions will be translated in the same way.

XXX Annotate this diagram with an arrow showing how the equilibrium solution changes.



Above to go from matrix to affine, we translate solutions by $\begin{bmatrix}2\\1\end{bmatrix}$. To go from affine to matrix, we translate solutions by $\begin{bmatrix}-2\\-1\end{bmatrix}$.

But, **make sure to actually compute the equilibrium solution(s)**. When confronted with an equation like $\vec{r}' = M\vec{r} + \vec{p}$, it is tempting to claim that $\vec{p}$ is an equilibrium solution. In all likelihood, **it is not**! (It is actually $-M^{-1}\vec{p}$.)

Practice Problems

1. Explain what you need to do in two different ways.
   (a) Way 1
   (b) Way 2
2. A second question.
3. A third question.

Solutions for Module 7

1. (a) Way 1 is good
   (b) Way 2 is better
2. The answer to the second question.
3.

47 Consider the following model of Social Media Usage where

$$P(t) = \text{millions of social media posts at year } \ t$$
$$U(t) = \text{millions of social media users at year } \ t$$

- (P1_P) Ignoring all else, each year posts decay proportionally to the current number of posts with proportionality constant 1.
- (P2_P) Ignoring all else (independent of decay), posts grow by a constant amount of 2 million posts every year.
- (P1_U) Ignoring all else, social media users increase/decrease in proportion to the number of posts.
- (P2_U) Ignoring all else, social media users increase/decrease in proportion to the number of users.
- (P3_U) Ignoring all else, 1 million people stop using the platform every year.

A school intervention is described by the parameter $a \in \left[-\frac{1}{2}, 1\right]$:

- After the intervention, the proportionality constant for (P1_U) is $1 - a$.
- After the intervention, the proportionality constant for (P2_U) is a.

47.1 Model this situation using a system of differential equations. Explain which parts of your model correspond to which premise(s).

48 The **SM** model of Social Media Usage is

$$P' = -P + 2$$
$$U' = (1-a)P + aU - 1$$

where

$$P(t) = \text{millions of social media posts at year } \ t$$
$$U(t) = \text{millions of social media users at year } \ t$$
$$a \in \left[-\frac{1}{2}, 1\right]$$

48.1 What are the equilibrium solution(s)?

48.2 Make a phase portrait for the system.

`https://www.desmos.com/calculator/h3wtwjghv0`

48.3 Use phase portraits to conjecture: what do you think happens to the equilibrium solution(s) as a transitions from negative to positive? Justify with a computation.

49 The **SM** model of Social Media Usage is

$$P' = -P + 2$$
$$U' = (1-a)P + aU - 1$$

where

$$P(t) = \text{millions of social media posts at year } \ t$$
$$U(t) = \text{millions of social media users at year } \ t$$
$$a \in \left[-\frac{1}{2}, 1\right]$$

49.1 Can you rewrite the system in matrix form? I.e., in the form $\vec{r}'(t) = M\vec{r}(t)$ for some matrix M where $\vec{r}(t) = \begin{bmatrix} P(t) \\ U(t) \end{bmatrix}$.

49.2 Define $\vec{s}(t) = \begin{bmatrix} S_{P(t)} \\ S_{U(t)} \end{bmatrix}$ to be the displacement from equilibrium in the **SM** model at time t (provided an equilibrium exists).

(a) Write $\vec{s}$ in terms of P and U.

(b) Find $\vec{s}'$ in terms of P and U.

(c) Find $\vec{s}'$ in terms of S_P and S_U.

(d) Can one of your differential equations for $\vec{s}$ be written in matrix form? Which one?

(e) Analytically classify the equilibrium solution for your differential equation for $\vec{s}$ when $a = -\frac{1}{2}$, $a = \frac{1}{2}$, and $a = 1$. (You may use a calculator for computing eigenvectors/values.)

50 The **SM** model of Social Media Usage is

$$P' = -P + 2$$
$$U' = (1-a)P + aU - 1$$

where

$$P(t) = \text{millions of social media posts at year } t$$
$$U(t) = \text{millions of social media users at year } t$$
$$a \in \left[-\frac{1}{2}, 1\right]$$

Some politicians have been looking at the model. They made the following posts on social media:

1. *The model shows the number of posts will always be increasing. SAD!*
2. *I see the number of social media users always increases. That's not what we want!*
3. *It looks like social media is just a fad. Although users initially increase, they eventually settle down.*
4. *I have a dream! That one day there will be social media posts, but eventually there will be no social media users!*

50.1 For each social media post, make an educated guess about what initial conditions and what value(s) of a the politician was considering.

50.2 The school board wants to limit the number of social media users to fewer than 10 million. Make a recommendation about what value of a they should target.

Linear Systems with Constant Coefficients (Complex Solutions)

In this module you will learn

- How to find explicit real solutions to systems of differential equations with constant coefficients when the eigenvalues of the coefficient matrix are complex.
- How the complex, distinct eigenvalues of a system of differential equations with constant coefficients are related to the stability of the equilibrium solutions.

Complex Numbers Pre-requisites

For this module, you a basic understanding of complex numbers. We assume you are familiar with:

- Adding, subtracting, multiplying and dividing complex numbers.
- The complex conjugate.
- Euler's formula.

Review these topics in Appendix G before reading this module.

Finding Explicit Solutions

Techniques from Module 6 and Module 7 can be used to find explicit solutions to matrix/affine differential equations so long as the coefficient matrix of the system has real eigenvalues.

A similar procedure can be used when the coefficient matrix has *complex* eigenvalues, though some care is needed if we are hoping for our explicit solutions to be *real* solutions.

Consider the system

$$\begin{aligned} x' &= y \\ y' &= -4x \end{aligned} \qquad \longleftrightarrow \qquad \vec{r}\,' = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} \vec{r} \qquad \text{where } \vec{r}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

Following Module 6, we can find explicit solutions by finding the eigenvalues and eigenvectors of $\begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}$. We find eigenvalues $\lambda_1 = 2i$ with eigenvector $v_1 = \begin{bmatrix} -i \\ 2 \end{bmatrix}$ and eigenvalue $\lambda_2 = -2i$ with eigenvector $v_2 = \begin{bmatrix} i \\ 2 \end{bmatrix}$.

Using these eigenvalues and eigenvectors, we can construct two linearly independent solutions to the system:

$$\vec{r}_1(t) = \begin{bmatrix} -i \\ 2 \end{bmatrix} e^{2ti} \tag{5}$$

$$\vec{r}_2(t) = \begin{bmatrix} i \\ 2 \end{bmatrix} e^{-2ti} \tag{6}$$

We can verify, by differentiating, that both $\vec{r}_1(t)$ and $\vec{r}_2(t)$ are solutions. Further, they are linearly independent, and so the general solution to the system of differential equations is

$$\vec{r}(t) = A\vec{r}_1(t) + B\vec{r}_2(t)$$

where A and B are parameters. The issue with this general solution is that its complex.

Finding Real Solutions

We know that the matrix equation

$$\vec{r}\,' = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} \vec{r}$$

has a general solution

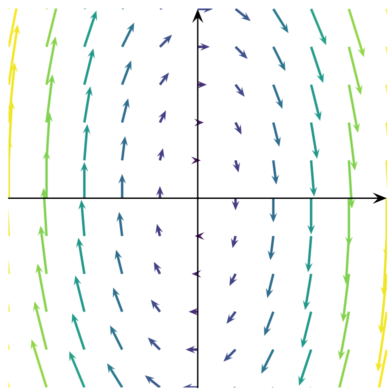
$$\vec{r}(t) = A\begin{bmatrix}-i\\2\end{bmatrix}e^{2ti} + B\begin{bmatrix}i\\2\end{bmatrix}e^{-2ti}. \tag{7}$$

where A and B are parameters. However, since the problem was posed in terms of a real equation (most likely coming from a model involving only real quantities), we would prefer a general solution that is real-valued.

Before continuing, let's convince ourselves that real solutions exist.

Looking at a phase portrait for the system, we can see that there should be real solutions (we could trace them with our fingers!).

Phase Portrait for
$\vec{r}' = \begin{bmatrix}0 & 1\\-4 & 0\end{bmatrix}\vec{r}$



Applying Euler's method to simulate solutions, we also see that our simulations are always real-valued.

The question now becomes, how can we manipulate Equation (7) to find real solutions? The answer relies on Euler's formula.

> **Theorem (Euler's Formula)**
>
> For any real number t we have:
>
> $$e^{it} = \cos(t) + i\sin(t)$$

Using Euler's formula, we can rewrite the solutions from Equation (5) and Equation (6) as

$$\begin{aligned}\vec{r}_1(t) = \begin{bmatrix}-i\\2\end{bmatrix}e^{2it} &= \begin{bmatrix}-i\\2\end{bmatrix}(\cos(2t) + i\sin(2t))\\ &= \begin{bmatrix}-i\cos(2t)\\2i\sin(2t)\end{bmatrix} = i\begin{bmatrix}-\cos(2t)\\2\sin(2t)\end{bmatrix}\end{aligned}$$

and

$$\vec{r}_2(t) = \begin{bmatrix} i \\ 2 \end{bmatrix} e^{-2it}$$

$$= \begin{bmatrix} i \\ 2 \end{bmatrix} (\cos(-2t) + i\sin(-2t))$$

$$= \begin{bmatrix} i \\ 2 \end{bmatrix} (\cos(2t) - i\sin(2t))$$

$$= \begin{bmatrix} i\cos(2t) \\ -2i\sin(2t) \end{bmatrix} = i\begin{bmatrix} \cos(2t) \\ -2\sin(2t) \end{bmatrix}.$$

Thus, the general solution from Equation (7) can be rewritten as

$$\vec{r}(t) = A\vec{r}_1(t) + B\vec{r}_2(t) = Ai\begin{bmatrix} -\cos(2t) \\ 2\sin(2t) \end{bmatrix} + Bi\begin{bmatrix} \cos(2t) \\ -2\sin(2t) \end{bmatrix}$$

$$= i\left(A\begin{bmatrix} -\cos(2t) \\ 2\sin(2t) \end{bmatrix} + B\begin{bmatrix} \cos(2t) \\ -2\sin(2t) \end{bmatrix}\right)$$

where A and B are parameters. Since we are allowed to pick A and B to be any value we like, we can try to pick A and B so that the resulting solution is real.

There are many options, but one such choice is to pick $A = -i$ and $B = 0$, which gives

$$\vec{r}_3(t) = i(-i)\begin{bmatrix} -\cos(2t) \\ 2\sin(2t) \end{bmatrix} = \begin{bmatrix} -\cos(2t) \\ 2\sin(2t) \end{bmatrix}$$

as a real solution. Alternatively, we could pick $A = 0$ and $B = -i$, which gives

$$\vec{r}_4(t) = i(-i)\begin{bmatrix} \cos(2t) \\ -2\sin(2t) \end{bmatrix} = \begin{bmatrix} \cos(2t) \\ -2\sin(2t) \end{bmatrix}$$

as a real solution. Since we now have two linearly-independent real solutions, we can write a general real solution

$$\vec{r}(t) = P\vec{r}_3(t) + Q\vec{r}_4(t) = P\begin{bmatrix} -\cos(2t) \\ 2\sin(2t) \end{bmatrix} + Q\begin{bmatrix} \cos(2t) \\ -2\sin(2t) \end{bmatrix}$$

where P and Q are parameters. Here, as long as we pick P and Q to be real numbers, the result will be a real solution.

Other Methods to Find Real Solutions

In the previous section, we exploited the general *complex* solution to find the general real solution. Specifically, we carefully chose the parameters in the general complex solution to find two linearly independent real solutions. Then, we took the linear combination of these two real solutions to find the general real solution. *This method always works*.[21] However, it is sometimes possible to save some time by using guess-and-check.

Consider again the general complex solution. After Equation (7) is expanded using Euler's formula, we have

$$\vec{r}(t) = Ai\begin{bmatrix} -\cos(2t) \\ 2\sin(2t) \end{bmatrix} + Bi\begin{bmatrix} \cos(2t) \\ -2\sin(2t) \end{bmatrix}.$$

We can then take the real and imaginary parts of this solution:

[21] That is, it works so long as there is a real solution

$$\vec{r}_{\text{real}}(t) = 0 \qquad \text{and} \qquad \vec{r}_{\text{imag}}(t) = A\begin{bmatrix} -\cos(2t) \\ 2\sin(2t) \end{bmatrix} + B\begin{bmatrix} \cos(2t) \\ -2\sin(2t) \end{bmatrix}.$$

Testing by differentiating, we can verify that $\vec{r}_{\text{real}}$ and $\vec{r}_{\text{imag}}$ are both real solutions to the original system. In fact, $\vec{r}_{\text{imag}}$ is the general real solution we already found!

Guessing-and-checking by examining the real and imaginary parts of a complex solution will often save you time—remember, all you need to write down a general real solution is the correct number of linearly independent real solutions.

Example. Find the general real solution to

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 0.11y$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = -11x$$

and the solution to the initial value problem $x(0) = 1$ and $y(0) = -1$.

XXX Finish

From the calculations above, we know that the general solution to the system of differential equations is

$$\vec{r}(t) = c_1\vec{s}_1(t) + c_2\vec{s}_2(t).$$

We only need to find the constants c_1 and c_2 that satisfy the initial conditions $\vec{r}(0) = \left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right]$. We have

$$\begin{aligned} \begin{bmatrix} 1 \\ -1 \end{bmatrix} &= \vec{r}(0) = c_1\vec{s}_1(0) + c_2\vec{s}_2(0) \\ &= c_1\begin{bmatrix} 2 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ -0.22 \end{bmatrix} \\ &= \begin{bmatrix} 2c_1 \\ -0.22c_2 \end{bmatrix} \end{aligned}$$

We get $c_1 = \frac{1}{2}$ and $c_2 = \frac{50}{11}$ and the solution is

$$\begin{aligned} \vec{r}(t) &= \left(\frac{1}{2}\right)\vec{s}_1(t) + \left(\frac{50}{11}\right)\vec{s}_2(t) \\ &= \begin{bmatrix} \cos(0.11t) - \left(\frac{50}{11}\right)\sin(0.11t) \\ -0.11\sin(0.11t) - \cos(0.11t) \end{bmatrix}. \end{aligned}$$

Stability of the Equilibrium Solutions

As we have seen in Module 6, the stability of equilibrium solutions of a system of differential equations with constant coefficients is determined by the eigenvalues of the coefficient matrix when the eigenvalues were real and distinct.

Similarly, when the eigenvalues are complex, the stability of the equilibrium solutions is also determined by the eigenvalues of the coefficient matrix. More specifically, the stability of the equilibrium solutions is determined by the *real part of the eigenvalues*.

Before reading the remainder of this module, we recommend that you gain some intuition about the solutions of systems of differential equations with complex eigenvalues by solving the core exercises in this module. XXX This paragraph should be moved to the margin.

Consider a system of differential equations

$$\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = M\vec{r},$$

where the eigenvalues of the coefficient matrix M are complex: $\lambda_1 = a + ib$ and $\lambda_2 = a - ib$.

First, observe that the unique equilibrium solution is $\vec{r}(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, so we want to establish the stability of this equilibrium solution.

Following the process described earlier in this module, we know that the solutions of the system of differential equations will have the form

$$\vec{r}_i(t) = \vec{v}_i e^{\lambda_i t} = \vec{v}_i e^{at} e^{ibt} = \vec{v}_i e^{at}(\cos(bt) + i\sin(bt)),$$

where $\vec{v}_i$ is the eigenvector associated with the eigenvalue λ_i.

We can now study what will happen to the solutions as t approaches infinity, by focusing on each term of the solution

$$\vec{r}_i(t) = \vec{v}_i e^{at}(\cos(bt) + i\sin(bt)).$$

The first term $\vec{v}_i$ is a constant vector, so it will not affect the stability of the equilibrium solution.

The third term $\cos(bt) + i\sin(at)$ is a periodic function, so it will not affect the stability of the equilibrium solution either.

The only term that can affect the stability of the equilibrium solution is the second term e^{at}.

- If $a < 0$, then e^{at} will approach 0 as t approaches infinity, and the equilibrium solution will be *stable* and *attracting*.
- If $a > 0$, then e^{at} will approach infinity as t approaches infinity, and the equilibrium solution will be *unstable* and *repelling*.
- If $a = 0$, then e^{at} will be equal to 1 for all t, and the equilibrium solution will be *stable* but neither attracting nor repelling.

Summary

Below we combine all the results obtained in the last modules about the stability of equilibrium solutions of systems of differential equations with constant coefficients.

Consider the following system of differential equations

$$\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = M\vec{r} + \vec{b},$$

where the eigenvalues of the coefficient matrix M are λ_1 and λ_2. The stability of the equilibrium solution $\vec{r}(t) = -M^{-1}\vec{b}$ is determined by the eigenvalues of the coefficient matrix M.

Eigenvalue λ_1	Eigenvalue λ_2	Stability of Equilibrium
$\lambda_1 < 0$	$\lambda_2 < 0$	Stable and Attracting
$\lambda_1 > 0$	$\lambda_2 > 0$	Unstable and Repelling
$\lambda_1 < 0$	$\lambda_2 > 0$	Unstable and Repelling
$\lambda_1 = 0$	$\lambda_2 < 0$	Stable

Eigenvalue λ_1	Eigenvalue λ_2	Stability of Equilibrium
$\lambda_1 = 0$	$\lambda_2 > 0$	Unstable
$\lambda_1 = a + ib, a < 0$	$\lambda_2 = a - ib, a < 0$	Stable and Attracting
$\lambda_1 = a + ib, a > 0$	$\lambda_2 = a - ib, a > 0$	Unstable and Repelling
$\lambda_1 = ib$	$\lambda_2 = -ib$	Stable

OR

Real/Complex	Sign of Eigenvalues	Stability of Equilibrium
Real	Both negative	Stable and Attracting
Real	Both positive	Unstable and Repelling
Real	One negative, one positive	Unstable and Repelling
Real	One is zero, one is negative	Stable
Real	One is zero, one is positive	Unstable
Complex	Real parts negative	Stable and Attracting
Complex	Real parts positive	Unstable and Repelling
Complex	Real parts zero	Stable

XXX CHOOSE ONE TABLE!!

Practice Problems

1. Explain what you need to do in two different ways.
   (a) Way 1
   (b) Way 2
2. A second question.
3. A third question.

Solutions for Module 8

1. (a) Way 1 is good
   (b) Way 2 is better
2. The answer to the second question.
3.

51 Consider the following **FD** model of Fleas and Dogs where

$$F(t) = \text{number of parasites (fleas) at year } \ t \ \ \text{(in millions)}$$
$$D(t) = \text{number of hosts (dogs) at year } \ t \ \ \text{(in thousands)}$$

- ($P1_F$) Ignoring all else, the number of parasites decays in proportion to its population (with constant 1).
- ($P2_F$) Ignoring all else, parasite numbers grow in proportion to the number of hosts (with constant 1).
- ($P1_D$) Ignoring all else, hosts numbers grow in proportion to their current number (with constant 1).
- ($P2_D$) Ignoring all else, host numbers decrease in proportion to the number of parasites (with constant 2).
- ($P1_c$) Anti-flea collars remove 2 million fleas per year.
- ($P2_c$) Constant dog breeding adds 1 thousand dogs per year.

51.1 Write a system of differential equations for the **FD** model.

51.2 Can you rewrite the system in matrix form $\vec{r}' = M\vec{r}$? What about in *affine* form $\vec{r}' = M\vec{r} + \vec{b}$?

51.3 Make a phase portrait for your model.

51.4 What should solutions to the system look like in the phase plane? What are the equilibrium solution(s)?

52 Recall the **FD** model of Fleas and Dogs where

$$F(t) = \text{number of parasites (fleas) at year } t \text{ (in millions)}$$
$$D(t) = \text{number of hosts (dogs) at year } t \text{ (in thousands)}$$
$$\vec{r}(t) = \begin{bmatrix} F(t) \\ D(t) \end{bmatrix}$$

and

$$\vec{r}'(t) = \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix} \vec{r}(t) + \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Define $\vec{s}(t)$ to be the displacement of $\vec{r}(t)$ from equilibrium at time t.

52.1 Find a formula for $\vec{s}$ in terms of $\vec{r}$.

52.2 Can you find a matrix M so that $\vec{s}'(t) = M\vec{s}(t)$?

52.3 What are the eigenvalues of M?

52.4 Find an eigenvector for each eigenvalue of M.

52.5 What are the eigen solutions for $\vec{s}' = M\vec{s}$?

53 Recall the **FD** model of Fleas and Dogs where

$$F(t) = \text{number of parasites (fleas) at year } t \text{ (in millions)}$$
$$D(t) = \text{number of hosts (dogs) at year } t \text{ (in thousands)}$$
$$\vec{r}(t) = \begin{bmatrix} F(t) \\ D(t) \end{bmatrix} \qquad \vec{s}(t) = \vec{r}(t) - \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

and

$$\vec{s}'(t) = M\vec{s}(t) \qquad \text{where} \qquad M = \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix}.$$

This equation has eigen solutions

$$\vec{s}_1(t) = e^{it}\begin{bmatrix} 1-i \\ 2 \end{bmatrix}$$
$$\vec{s}_2(t) = e^{-it}\begin{bmatrix} 1+i \\ 2 \end{bmatrix}.$$

53.1 Recall Euler's formula $e^{it} = \cos(t) + i\sin(t)$.

(a) Use Euler's formula to expand $\vec{s}_1 + \vec{s}_2$. Are there any imaginary numbers remaining?

(b) Use Euler's formula to expand $i(\vec{s}_1 - \vec{s}_2)$. Are there any imaginary numbers remaining?

53.2 Verify that your formulas for $\vec{s}_1 + \vec{s}_2$ and $i(\vec{s}_1 - \vec{s}_2)$ are solutions to $\vec{s}'(t) = M\vec{s}(t)$.

53.3 Can you give a third *real* solution to $\vec{s}'(t) = M\vec{s}(t)$?

54 Recall the **FD** model of Fleas and Dogs where

$$F(t) = \text{number of parasites (fleas) at year } \; t \; \text{ (in millions)}$$
$$D(t) = \text{number of hosts (dogs) at year } \; t \; \text{ (in thousands)}$$
$$\vec{r}(t) = \begin{bmatrix} F(t) \\ D(t) \end{bmatrix} \qquad \vec{s}(t) = \vec{r}(t) - \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

and

$$\vec{s}\,'(t) = M\vec{s}(t) \qquad \text{where} \qquad M = \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix}.$$

54.1 What is the dimension of the space of solutions to $\vec{s}\,'(t) = M\vec{s}(t)$?

54.2 Give a basis for all solutions to $\vec{s}\,'(t) = M\vec{s}(t)$.

54.3 Find a solution satisfying $\vec{s}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

54.4 Using what you know, find a general formula for $\vec{r}(t)$.

54.5 Find a formula for $\vec{r}(t)$ satisfying $\vec{r}(0) = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$.

55 Recall the **FD** model of Fleas and Dogs where

$$F(t) = \text{number of parasites (fleas) at year } t \text{ (in millions)}$$
$$D(t) = \text{number of hosts (dogs) at year } t \text{ (in thousands)}$$
$$\vec{r}(t) = \begin{bmatrix} F(t) \\ D(t) \end{bmatrix} \qquad \vec{s}(t) = \vec{r}(t) - \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

and

$$\vec{s}\,'(t) = M\vec{s}(t) \qquad \text{where} \qquad M = \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix}.$$

Some research is being done on a shampoo for the dogs. It affects flea and dog reproduction:

- (PS_F) Ignoring all else, the number of parasites decays in proportion to its population with constant $1 + a$.
- (PS_D) Ignoring all else, hosts numbers grow in proportion to their current number with constant $1 - a$.
- $-1 \leq a \leq 1$.

These premises replace (P1_F) and (P1_D).

55.1 Modify the previous **FD** model to incorporate the effects of the shampoo.

55.2 Make a phase portrait for the **FD Shampoo** model.

55.3 Find the equilibrium solutions for the **FD Shampoo** model.

55.4 For each equilibrium solution determine its stability/instability/etc.

55.5 Analytically justify your conclusions about stability/instability/etc.

56 Consider the differential equation

$$\vec{s}'(t) = M\vec{s}(t) \quad \text{where} \quad M = \begin{bmatrix} -1 & -4 \\ 2 & 3 \end{bmatrix}$$

56.1 Make a phase portrait. Based on your phase portrait, classify the equilibrium solution.

`https://www.desmos.com/calculator/h3wtwjghv0`

56.1 Find eigen solutions for this differential equation (you may use a calculator/computer to assist).

56.2 Find a general *real* solution.

56.3 Analytically classify the equilibrium solution.

Quantitative Analysis: Linearization

In this module you will learn

- How to use *linearization*, the process of finding an affine approximation to a system of differential equations, to analyze the equilibrium solutions of non-linear systems of differential equations.
- When using linearization to study equilibrium solutions is appropriate.

We have a complete theory, based on eigenvalues, for classifying the equilibria of systems of differential equations that can be written in matrix or affine form. Unfortunately, most equations that come from real-world models cannot be written in matrix or affine form!

Linearization

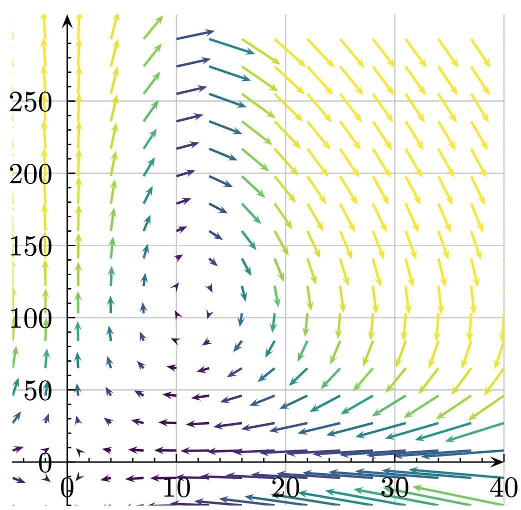
Recall the Fox and Rabbit model from Core Exercise 17:

$$F'(t) = 0.01 \cdot R(t) \cdot F(t) - 1.1 \cdot F(t)$$
$$R'(t) = 1.1 \cdot R(t) - 0.1 \cdot F(t) \cdot R(t)$$

where $F(t)$ is the population of foxes (in millions) at year t and $R(t)$ is the population of rabbits (in millions) at year t.

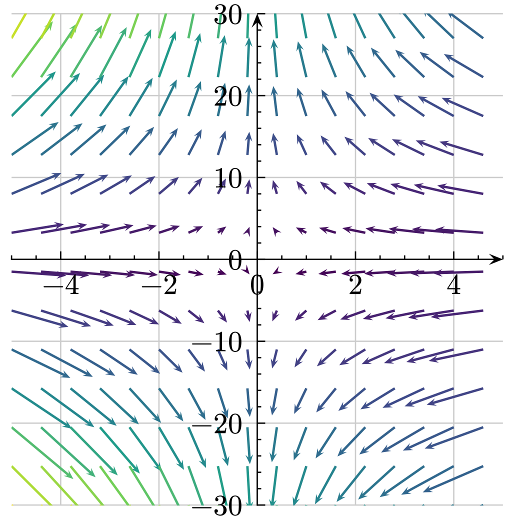
Based on the phase portrait, there appears to be an unstable equilibrium at $(F, R) = (0, 0)$ and a stable equilibrium at $(F, R) = (11, 110)$.

Foxes vs. Rabbits



We cannot use eigenvalue analysis to classify the equilibrium solutions because non-linear systems do not have eigenvalues. However, we may be able to find a matrix/affine system that is “close to” the Fox-and-Rabbit model and apply eigen analysis to our approximation.

Let’s focus on the equilibrium $(F, R) = (0, 0)$. Zooming in, the phase portrait looks familiar—very similar to the phase portrait for a matrix equation.



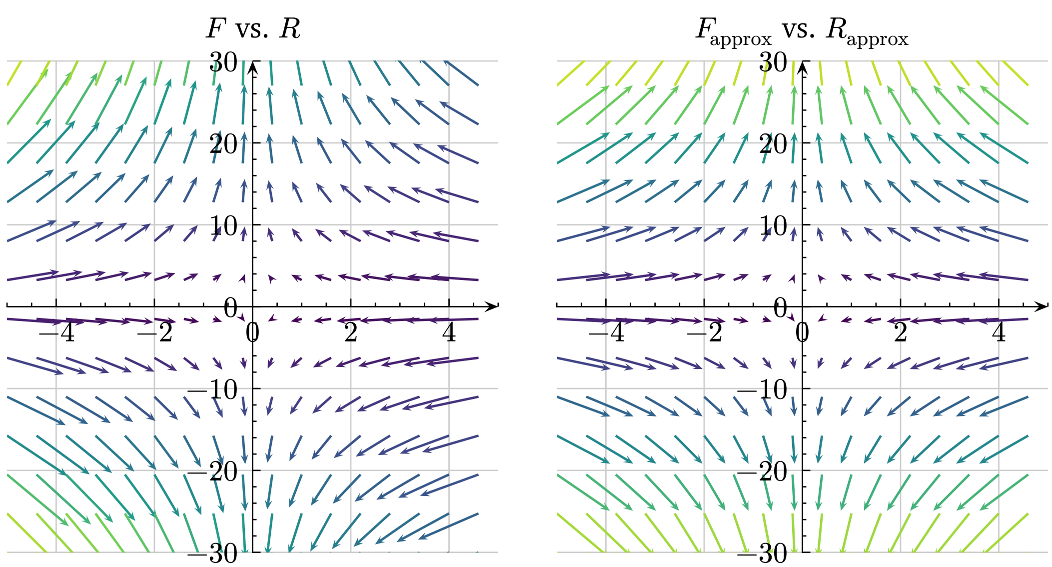
We will try to figure out an approximating matrix equation via hand-wavy methods first. A useful heuristic is that if a quantity is small, then that quantity squared is very small and can be ignored.[22] Recall our system:

$$F' = 0.01 \cdot F \cdot R - 1.1 \cdot F$$
$$R' = 1.1 \cdot R - 0.1 \cdot F \cdot R$$

We are studying the behaviour when $F \approx 0$ and $R \approx 0$, so $F \cdot R$ is very, very small. Replacing all $F \cdot R$ terms with 0 gives us

$$\begin{aligned} F'_{\text{approx}} &= -1.1 \cdot F_{\text{approx}} \\ R'_{\text{approx}} &= 1.1 \cdot R_{\text{approx}} \end{aligned} \qquad \iff \qquad \begin{bmatrix} F_{\text{approx}} \\ R_{\text{approx}} \end{bmatrix}' = \begin{bmatrix} -1.1 & 0 \\ 0 & 1.1 \end{bmatrix} \begin{bmatrix} F_{\text{approx}} \\ R_{\text{approx}} \end{bmatrix}$$

Graphing phase portraits side-by-side, the original system and our approximating system look very similar near $(F, R) = (0, 0)$.



If we simulated solutions, we would find that $F(t) \approx F_{\text{approx}}(t)$ and $R(t) \approx R_{\text{approx}}(t)$, when both $F(t)$ and $R(t)$ are near 0. Since F_{approx} and R_{approx} can be analyzed using eigen techniques (eigenvalues of -1.1 and 1.1 mean $(0, 0)$ is *unstable*), and since F and R behave similarly to F_{approx} and R_{approx} near $(0, 0)$, we conclude that $(F, R) = (0, 0)$ is an unstable equilibrium.

[22] Look back at proofs for the derivative of x^2; using limits formalizes the rule, but all higher-order terms go to zero in the limit.

The process of finding a matrix/affine system that closely approximates a non-linear system near a point is called *linearization*.

Using Calculus to Linearize

In the previous example, we guessed our way into a linearization. But we have a more systematic tool at our disposal: *Calculus*.

Recall from Calculus that for a function $f:\mathbb{R}\to\mathbb{R}$, the tangent line to the graph $y=f(x)$ at the point $(E,f(E))$ is given by

$$y=f(E)+f'(E)(x-E).$$

That means that when $x\approx E$, we have

$$f(x) \quad\approx\quad f(E)+f'(E)(x-E). \tag{8}$$

There is a similar formula for multi-variable functions.[23] Let $\vec{F}(x,y)=\begin{bmatrix}F_1(x,y)\\F_2(x,y)\end{bmatrix}$ and let $\vec{E}\in\mathbb{R}^2$. Then, when $\begin{bmatrix}x\\y\end{bmatrix}\approx\vec{E}$,

$$\vec{F}(x,y) \quad\approx\quad \vec{F}\left(\vec{E}\right)D_{\vec{F}}\left(\vec{E}\right)\left(\begin{bmatrix}x\\y\end{bmatrix}-\vec{E}\right). \tag{9}$$

Here, $D_{\vec{F}}\left(\vec{E}\right)$ is the *total derivative* (also called the *Jacobian matrix* or *Jacobian*) of $\vec{F}$ at $\vec{E}$. That is,

$$D_{\vec{F}}\left(\vec{E}\right)=\begin{bmatrix}\frac{\partial F_1}{\partial x}&\frac{\partial F_1}{\partial y}\\\frac{\partial F_2}{\partial x}&\frac{\partial F_2}{\partial y}\end{bmatrix}\qquad\text{evaluated at }\vec{E}.$$

Approximations like in Equations (8) and (9) called *affine approximations* or *first-order approximations*.

Example. Let $\vec{F}(x,y)=(x^2y,y^3-2)$ and let $\vec{E}=(1,1)$. Find an affine approximation to $\vec{F}$ at $\vec{E}$.

XXX Finish

Affine approximations are exactly the tool we need, since they turn non-linear functions into affine ones. Let's apply affine approximations to the Fox-and-Rabbit model.

Recall:

$$F'(t)=0.01\cdot R(t)\cdot F(t)-1.1\cdot F(t)$$
$$R'(t)=1.1\cdot R(t)-0.1\cdot F(t)\cdot R(t)$$

Define $\vec{G}(x,y)=\begin{bmatrix}0.01xy-1.1x\\1.1y-0.1xy\end{bmatrix}$. Our equation can now be written as

$$\begin{bmatrix}F'(t)\\R'(t)\end{bmatrix}=\vec{G}(F(t),R(t)).$$

It's important to note here that when we write F' or R', we mean $\frac{\mathrm{d}F(t)}{\mathrm{d}t}$ or $\frac{\mathrm{d}R(t)}{\mathrm{d}t}$, i.e., the derivative with respect to t. The function $\vec{G}$, on the other hand, is a function of two variables and helps us re-write our differential equation but is not directly related to the variable t.

To approximate our differential equation, we need to find the total derivative of $\vec{G}$.

[23] Here, we give the formula for functions $\vec{F}:\mathbb{R}\to\mathbb{R}^2$, but the generalized formula also holds for functions $\vec{F}:\mathbb{R}\to\mathbb{R}^n$.

$$D_{\vec{G}}(x,y) = \begin{bmatrix} \frac{\partial}{\partial x}(0.01xy - 1.1x) & \frac{\partial}{\partial y}(0.01xy - 1.1x) \\ \frac{\partial}{\partial x}(1.1y - 0.1xy) & \frac{\partial}{\partial y}(1.1y - 0.1xy) \end{bmatrix} = \begin{bmatrix} 0.01y - 1.1 & 0.01x \\ -0.1y & -0.1x + 1.1 \end{bmatrix}$$

Approximating near the equilibrium $(F, R) = (0, 0)$, we have

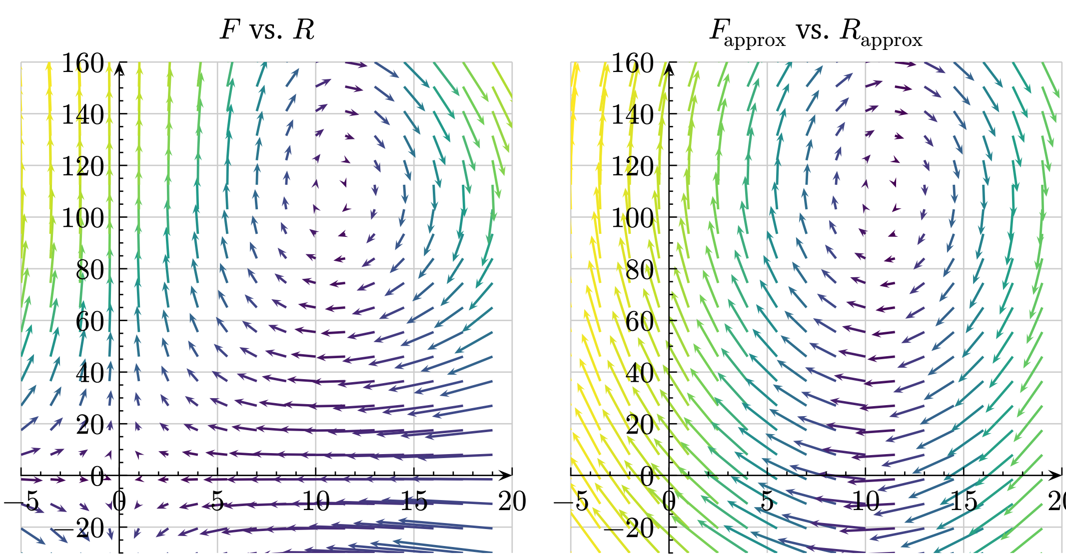
$$\begin{aligned} \begin{bmatrix} F'_{\text{approx}} \\ R'_{\text{approx}} \end{bmatrix} &= \vec{G}(0,0) + D_{\vec{G}}(0,0)\left(\begin{bmatrix} F_{\text{approx}} \\ R_{\text{approx}} \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right) \\ &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -1.1 & 0 \\ 0 & 1.1 \end{bmatrix} \begin{bmatrix} F_{\text{approx}} \\ R_{\text{approx}} \end{bmatrix} \\ &= \begin{bmatrix} -1.1 & 0 \\ 0 & 1.1 \end{bmatrix} \begin{bmatrix} F_{\text{approx}} \\ R_{\text{approx}} \end{bmatrix}, \end{aligned}$$

which is the same formula we found before.

We can also linearize near the equilibrium $(F, R) = (11, 110)$.

$$\begin{aligned} \begin{bmatrix} F'_{\text{approx}} \\ R'_{\text{approx}} \end{bmatrix} &= \vec{G}(11,110) + D_{\vec{G}}(11,110)\left(\begin{bmatrix} F_{\text{approx}} \\ R_{\text{approx}} \end{bmatrix} - \begin{bmatrix} 11 \\ 110 \end{bmatrix} \right) \\ &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0.11 \\ -11 & 0 \end{bmatrix} \left(\begin{bmatrix} F_{\text{approx}} \\ R_{\text{approx}} \end{bmatrix} - \begin{bmatrix} 11 \\ 110 \end{bmatrix} \right) \\ &= \begin{bmatrix} 0 & 0.11 \\ -11 & 0 \end{bmatrix} \begin{bmatrix} F_{\text{approx}} \\ R_{\text{approx}} \end{bmatrix} + \begin{bmatrix} -12.1 \\ 121 \end{bmatrix}. \end{aligned}$$

Comparing phase portraits, we see they are quite similar **near the point $(F, R) = (11, 110)$.**



Linearized Solutions vs. True Solutions

Solutions to a linearized system closely approximate solutions to the original system *when the values of the solution are near to the point of linearization*.

Continuing with the Fox-and-Rabbit example, let's focus on the linearization near $(F, R) = (11, 110)$. We have the original model

$$\begin{aligned} F' &= 0.01 \cdot F \cdot R - 1.1 \cdot F \\ R' &= 1.1 \cdot R \qquad - 0.1 \cdot F \cdot R \end{aligned}$$

and the approximate model

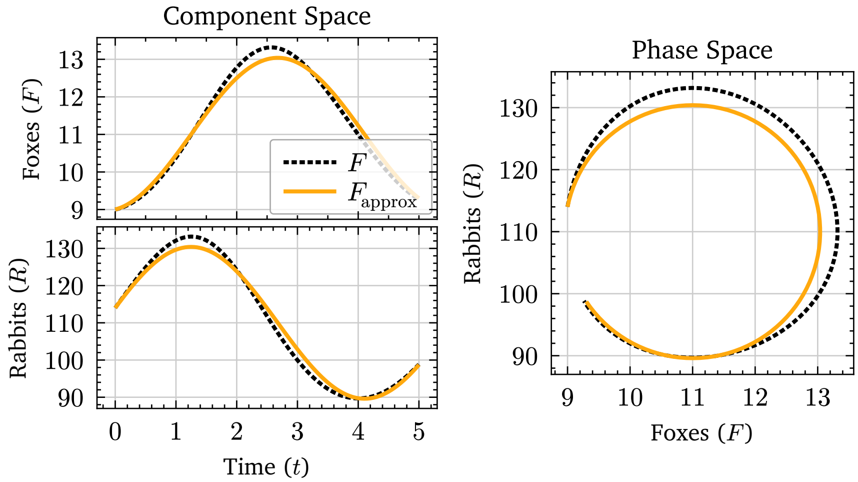
$$F'_{\text{approx}} = 0.11 \cdot R_{\text{approx}} - 12.1$$
$$R'_{\text{approx}} = -11 \cdot F_{\text{approx}} + 121.0$$

Using initial conditions of $F \approx 11$ and $R \approx 110$, we can compare solutions to the two systems.

$(F, R) = (9, 114)$

Approximate solution (solid) and true solution (dashed) with initial conditions

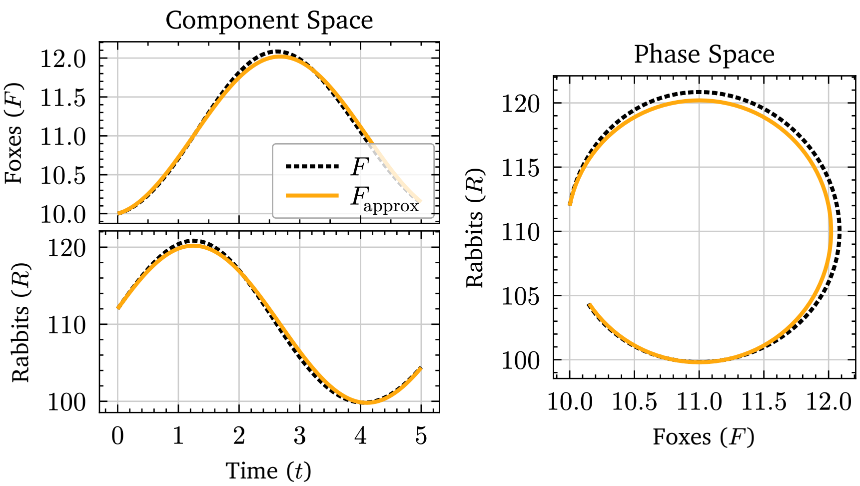
$$F(0) = 9$$
$$R(0) = 114$$



$(F, R) = (10, 112)$

Approximate solution (solid) and true solution (dashed) with initial conditions

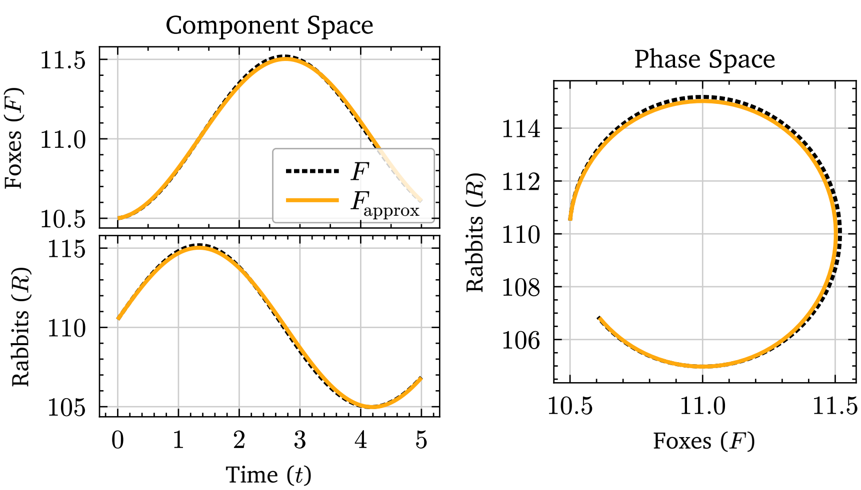
$$F(0) = 10$$
$$R(0) = 112$$



$(F, R) = (10.5, 110.5)$

Approximate solution (solid) and true solution (dashed) with initial conditions

$$F(0) = 10.5$$
$$R(0) = 110.5$$



In the figures, you can see that as the initial conditions get closer to $(F, R) = (10, 110)$ (where we centered our approximation), the solution to the linearized system (the solid curve) matches more an more closely to the solution to the original system (the dashed curve).[24]

Classifying Equilibria using Linearization

For suitably nice differential equations, linearization can be used to classify equilibrium solutions so long as corresponding equilibrium solution in the linearized system is **attracting or repelling**.

Theorem (Classification via Linearization)

Suppose $\vec{G}:\mathbb{R}^n\to\mathbb{R}^n$ is differentiable and consider the system of differential equations defined by

$$\vec{F}' = \vec{G}\big(\vec{F}\big). \tag{10}$$

Let $\vec{F}(t)=\vec{E}$ be an equilibrium solution to Equation (10) and let $\vec{F}'_{\text{approx}} = \ldots$ be the linearization of Equation (10) centered at $\vec{E}$.

Then, the following hold.

- If $\vec{F}_{\text{approx}}(t)=\vec{E}$ is an **attracting** equilibrium solution then $\vec{F}(t)=\vec{E}$ is an **attracting** equilibrium solution.
- If $\vec{F}_{\text{approx}}(t)=\vec{E}$ is an **repelling** equilibrium solution then $\vec{F}(t)=\vec{E}$ is an **repelling** equilibrium solution.
- If $\vec{F}_{\text{approx}}(t)=\vec{E}$ is neither attracting nor repelling, the nature of $\vec{F}(t)=\vec{E}$ cannot be determined from the linearization.

Why is the above theorem limited? What can go wrong in the case of stable/unstable equilibria that are not attracting/repelling?

Identical Linearization for Different Systems

Consider the following two systems of differential equations:

$$\begin{aligned} x'_{\text{lin}} &= -y_{\text{lin}} \\ y'_{\text{lin}} &= x_{\text{lin}} \end{aligned} \tag{11}$$

and

$$\begin{aligned} x'_{\text{nonlin}} &= -y_{\text{nonlin}} - x^3_{\text{nonlin}} \\ y'_{\text{nonlin}} &= x_{\text{nonlin}} \end{aligned} \tag{12}$$

The both have an equilibrium solutions $(x(t),y(t))=(0,0)$. Since Equation (11) can be rewritten as a matrix equation, we can solve it explicitly:

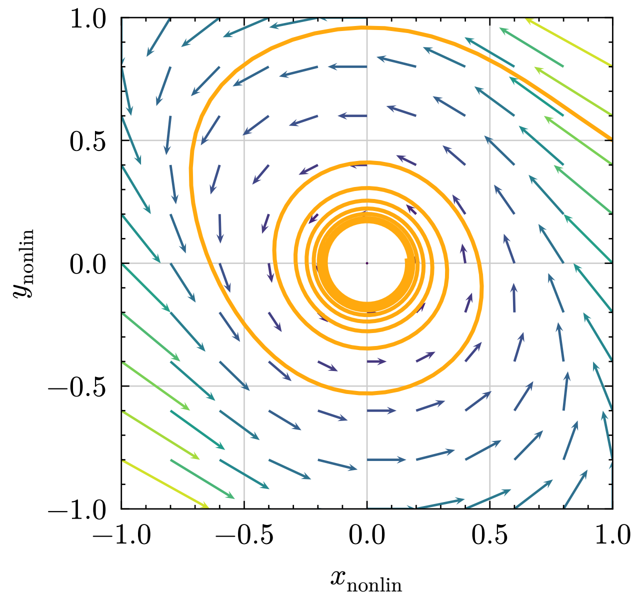
$$\begin{aligned} x_{\text{lin}}(t) &= A\cos(t) - B\sin(t) \\ y_{\text{lin}}(t) &= A\sin(t) + B\cos(t) \end{aligned} \qquad \text{where } A,B\in\mathbb{R} \text{ are parameters.}$$

From this, we see that the equilibrium solution at the origin for Equation (11) is *stable* and *not attracting*.

Unfortunately, Equation (12) is not so easily solved, so we use other tools.

[24]You might wonder if the reduction in error is meaningful. Since as $(F,R)\to(10,110)$, solutions look more and more constant, there is less oscillation overall when initial conditions are close to $(10,110)$. Thus, we would expect approximations (provided that they also oscillate less) to more closely match the original system. This is certainly true and is an observation about *absolute error*. However, if you pay attention to the scale on the axes of the figures, you see that the *relative error* is decreasing. That is, even if you zoom in (negating the lessening of oscillations), the approximation still more closely matches the true solution.

Looking at a phase portrait for Equation (12), it looks like the arrows might be spiralling in, but it is hard to tell. Overlaying with a simulated solution, the solution appears to spiral towards the origin.



A phase portrait for Equation (12) overlayed with a simulated solution starting at $x_{\text{nonlin}}(0) = 1$ and $y_{\text{nonlin}}(0) = \frac{1}{2}$.

We can make our analysis rigorous by analyzing Equation (12) directly. If solutions spiral towards the origin, then there should consistently be a component of $\begin{bmatrix} x'_{\text{nonlin}} \\ y'_{\text{nonlin}} \end{bmatrix}$ that points towards the origin. Since, at the point $\begin{bmatrix} x \\ y \end{bmatrix}$, the vector $\begin{bmatrix} -x \\ -y \end{bmatrix}$ points towards the origin, we can use linear algebra to find the component of $\begin{bmatrix} x'_{\text{nonlin}} \\ y'_{\text{nonlin}} \end{bmatrix} = \begin{bmatrix} -y-x^3 \\ x \end{bmatrix}$ that points in the direction of $\begin{bmatrix} -x \\ -y \end{bmatrix}$. After computing, we see

$$\begin{bmatrix} x'_{\text{nonlin}} \\ y'_{\text{nonlin}} \end{bmatrix} = \frac{x^4}{\sqrt{x^2+y^2}} \begin{bmatrix} -x \\ -y \end{bmatrix} + \frac{yx^3+x^2+y^2}{\sqrt{x^2+y^2}} \begin{bmatrix} -y \\ x \end{bmatrix},$$

and so, as long as $x \neq 0$, the vector $\begin{bmatrix} x'_{\text{nonlin}} \\ y'_{\text{nonlin}} \end{bmatrix}$ points towards the origin (and never points away from the origin). Since $x = 0$ and $y \neq 0$ is not an equilibrium solution, we conclude that all non-equilibrium solutions spiral towards the origin.

The conclusion is that for Equation (12), the equilibrium solution is *stable* and *attracting*.

Now, let's see what linearization tells us. It turns out that linearizing Equation (11) and Equation (12) at the origin both result in the same formula.

$$\begin{aligned} x'_{\text{approx}} &= -y_{\text{approx}} \\ y'_{\text{approx}} &= x_{\text{approx}} \end{aligned} \tag{13}$$

The equilibrium solution for Equation (13) is *stable* and *not attracting/repelling*. We can summarize what we've just learned in a table.

Equation	Equilibrium Classification	Equilibrium of *Linearization*
$x'_{\text{lin}} = -y_{\text{lin}}$ $y'_{\text{lin}} = x_{\text{lin}}$	stable, **not** attracting/repelling	stable, **not** attracting/repelling

Equation	Equilibrium Classification	Equilibrium of *Linearization*
$x'_{\text{nonlin}} = -y_{\text{nonlin}} - x^3_{\text{nonlin}}$ $y'_{\text{nonlin}} = x_{\text{nonlin}}$	stable, attracting	stable, **not** attracting/repelling

From this example, we can see that if the linearization of a system of differential equations has a stable equilibrium that is not attracting nor repelling, we don't know much about the original system. Its corresponding equilibrium could be stable and attracting, stable and not attracting, or unstable!

XXX Add part about where solutions to the linearized system drift out of phase with the original?

Practice Problems

1. Explain what you need to do in two different ways.
   (a) Way 1
   (b) Way 2
2. A second question.
3. A third question.

Solutions for Module 9

1. (a) Way 1 is good
   (b) Way 2 is better
2. The answer to the second question.
3.

57 Recall the tree model from Core Exercise 28:

- $H(t)$ = height (in meters) of tree trunk at time t
- $A(t)$ = surface area (in square meters) of all leaves at time t

$$H'(t) = 0.3 \cdot A(t) - b \cdot H(t)$$

$$A'(t) = -0.3 \cdot (H(t))^2 + A(t)$$

and $0 \leq b \leq 2$

A phase portrait for this model is available at

`https://www.desmos.com/calculator/tvjag852ja`

57.1 Visually classify the stability of each equilibrium solution as attracting/repelling/etc. Does the stability depend on b? Are you confident in your visual assessment?

57.2 Can you rewrite the system in matrix/affine form? Why or why not?

58 A simple logistic model for a population is

$$\frac{\mathrm{d}P}{\mathrm{d}t} = P(t) \cdot \left(1 - \frac{P(t)}{2}\right)$$

where $P(t)$ represents the population at time t.

We'd like to approximate $\frac{\mathrm{d}P}{\mathrm{d}t}$ when $P \approx \frac{1}{2}$.

58.1 What is the value of $\frac{\mathrm{d}P}{\mathrm{d}t}$ when $P = \frac{1}{2}$?

58.2 Define $f(P) = P \cdot \left(1 - \frac{P}{2}\right)$ and notice $\frac{\mathrm{d}P}{\mathrm{d}t} = f(P(t))$.

Approximate $\frac{\mathrm{d}P}{\mathrm{d}t}$ (i.e, approximate f) when $P = \frac{1}{2} + \Delta$ and Δ is small.

58.3 Write down an approximation $S(\Delta)$ that approximates $\frac{\mathrm{d}P}{\mathrm{d}t}$ when P is Δ away from $\frac{1}{2}$.

58.4 Let $A_{\frac{1}{2}}(P)$ be an *affine* approximation to $\frac{\mathrm{d}P}{\mathrm{d}t}$ that is a good approximation when $P \approx \frac{1}{2}$. Find a formula for $A_{\frac{1}{2}}(P)$.

58.5 Find additional affine approximations to $\frac{\mathrm{d}P}{\mathrm{d}t}$ centered at each equilibrium solution.

59 Based on our calculations from Core Exercise 58, we have several different affine approximations.

(Original)	$P' = P\left(1 - \frac{P}{2}\right)$	(https://www.desmos.com/calculator/v1coz4shtw)
$(A_{\{\frac{1}{2}\}})$	$P' \approx \frac{3}{8} + \frac{1}{2}\left(P - \frac{1}{2}\right)$	(https://www.desmos.com/calculator/zsb2apxhqs)
(A_0)	$P' \approx P$	(https://www.desmos.com/calculator/vw48bvqgrc)
(A_2)	$P' \approx -(P-2)$	(https://www.desmos.com/calculator/i2utk6vnqh)

59.1 What are the similarities/differences in the Desmos plots of solutions to the original equation vs. the other equations?

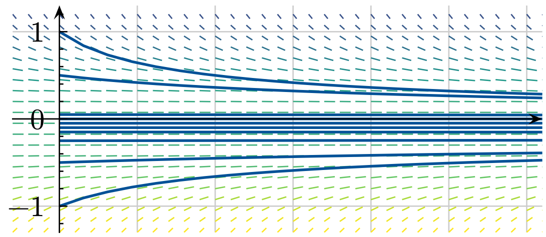
59.2 Does the nature of the equilibrium solutions change when using an affine approximation?

59.3 Classify each equilibrium solution of the original equation by using affine approximations.

60 Consider the differential equation whose slope field is sketched below.

$$P'(t) = -P(t) \cdot (0.1 + P(t)) \cdot (0.2 + P(t))$$
$$= -(P(t))^3 - 0.3 \cdot (P(t))^2 - 0.02 \cdot P(t)$$

https://www.desmos.com/calculator/ikp9rgo0kv



60.1 Find all equilibrium solutions.

60.2 Use affine approximations to classify the equilibrium solutions as stable/unstable/etc.

61 To make a 1d affine approximation of a function f at the point E we have the formula

$$f(x) \qquad \approx \qquad f(E) + f'(E)(x - E).$$

To make a 2d approximation of a function $\vec{F}(x, y) = (F_1(x, y), F_2(x, y))$ at the point $\vec{E}$, we have a similar formula

$$\vec{F}(x, y) \qquad \approx \qquad \vec{F}\left(\vec{E}\right) + D_{\vec{F}}\left(\vec{E}\right)\left(\begin{bmatrix} x \\ y \end{bmatrix} - \vec{E}\right)$$

where $D_{\vec{F}}\left(\vec{E}\right)$ is the *total derivative* of $\vec{F}$ at $\vec{E}$, which can be expressed as the matrix

$$D_{\vec{F}}\left(\vec{E}\right) = \begin{bmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{bmatrix}$$

evaluated at $\vec{E}$.

Recall our model from Exercise Core Exercise 28 for the life cycle of a tree where $H(t)$ was height, $A(t)$ was the leaves' surface area, and t was time:

$$H'(t) = 0.3 \cdot A(t) - b \cdot H(t)$$
$$A'(t) = -0.3 \cdot (H(t))^2 + A(t)$$

with $0 \leq b \leq 2$

We know the following:

- The equations cannot be written in matrix form.
- The equilibrium points are $(0, 0)$ and $\left(\frac{100}{9}b, \frac{1000}{27}b^2\right)$.

We want to find an affine approximation to the system.

Define $\vec{F}(H, A) = (H', A')$

61.1 Find the matrix for $D_{\vec{F}}$, the total derivative of $\vec{F}$.

61.2 Create an affine approximation to $\vec{F}$ around $\vec{e} = (0, 0)$ and use this to write an approximation to the original system.

61.3 In the original system, the equilibrium $(0, 0)$ is unstable and not repelling. Justify this using your affine approximation.

61.4 Create an affine approximation to $\vec{F}$ around $\vec{e} = \left(\frac{100}{9}b, \frac{1000}{27}b^2\right)$ and use this to write an approximation to the original system.

61.5 Make a phase portrait for the original system and your approximation from part 61.4. How do they compare?

61.6 Analyze the nature of the equilibrium solution in part 61.4 using eigen techniques. Relate your analysis to the original system.

62 Define $\vec{F}(x,y) = \begin{bmatrix} y \\ -xy+x^2-x-y \end{bmatrix}$ and consider the differential equation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \vec{F}(x,y).$$

62.1 Make a phase portrait for this differential equation. Based on your phase portrait, can you determine the nature of the equilibrium at $(0,0)$?

`https://www.desmos.com/calculator/peby3xd7jj`

62.2 Find an affine approximation to $\vec{F}$ centered at $(0,0)$.

62.3 Write down a differential equation that approximates the original equation near $(0,0)$.

62.4 Analyze the nature of the equilibrium solution $\vec{r}(t) = (0,0)$ using eigen techniques. (You may use a computer to assist in eigen computations.) Relate your analysis to the original system.

Higher Order Differential Equations

In this module you will learn

- How to use transform a higher order differential equation into a first order system of differential equations
- How to analyze the stability of a higher order differential equation using eigenvalues and eigenvectors
- How to approximate solutions to higher order differential equations using numerical methods

Order of a differential equation

Order of a Differential Equation. A differential equation is said to be ***order*** n if the highest derivative that appears in the differential equation is the nth derivative.

Up to this module, we only studied first order differential equations (and systems).

In this module, we are going to study higher order differential equations, which are differential equations that involve derivatives of order greater than one.

Transforming a higher order differential equation into a first order system of differential equations

When studying a higher order differential equation, we can rewrite it as an equivalent system of first order differential equations by introducing an auxiliary variable for each of the derivatives (except the highest one).

Example. Write the third order differential equation

$$u'''(x) - \tan(u''(x)) + 3\sqrt{1 + (u'(x))^2} + \sin(u''(x)) \cdot u(x) = e^x$$

as a system of first order differential equations.

We introduce a new variable for each derivative (except the highest one):

- $v(x) = u'(x)$;
- $w(x) = v'(x)$, and notice that $w(x) = u''(x)$, so $w'(x) = u'''(x)$.

We can then write the differential equation using these variables and their derivatives of order at most one:

$$w' - \tan(w) + 3\sqrt{1 + v^2} + \sin(w) \cdot u = e^x$$

Thus the original third order differential equation is equivalent to

$$\begin{cases} u' &= v \\ v' &= w \\ w' &= \tan(w) - 3\sqrt{1 + v^2} - \sin(w) \cdot u + e^x \end{cases}$$

This means that all the machinery that we developed in the previous modules for systems of differential equations applies directly to higher-order differential equations.

Equilibrium solutions for higher order differential equations

The definition of *equilibrium solution* remains the same, which means that an equilibrium solution will satisfy

$$u(x) \equiv C$$
$$u'(x) \equiv 0$$
$$u''(x) \equiv 0$$
$$\vdots$$

for some constant C.

So when we transform a higher order differential equation into a system, the equilibrium becomes a vector with a very specific form

$$\vec{r} = \begin{bmatrix} C \\ 0 \\ 0 \\ \vdots \end{bmatrix}$$

Stability of equilibrium solutions of a higher order differential equation

Recall the definitions introduced in Module 4.

To study the stability, we transform the equation into a system and study its eigenvalues according to the table at the end of Module 8.

Example. What is the stability of the equilibrium solutions of the differential equation $u''(x) = -u(x)$?

The first step is to find the equilibrium solution which has zero derivatives, so it satisfies

$$0 = -u$$

and so it has one equilibrium solution $u(x) = 0$.

The second step is to introduce an auxiliary variable

- $v(x) = u'(x)$

We can then write the differential equation as

$$\begin{cases} u' &= v \\ v' &= -u \end{cases}$$

We introduce a new variable $\vec{r}(x) = \begin{bmatrix} u(x) \\ v(x) \end{bmatrix}$, and we can write the differential equation as a system in matrix form

$$\vec{r}' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \vec{r}$$

This matrix has the eigenvalues $\pm i$, so we know that the equilibrium solution $\vec{r} = 0$ is stable but not attracting nor repelling.

Because the system is equivalent to the original differential equation, we can conclude that the equilibrium solution $u = 0$ is also stable but neither attracting nor repelling.

Linearization of higher order differential equations

The linearization of a higher order differential equation is done in the same way as for first order differential equations, by transforming it into a system and then linearizing the system.

It is important to re-write the differential equation as a system of first order differential equations before linearizing it. It is easy to make mistakes if we try to linearize a higher order differential equation directly.

Numerical methods for higher order differential equations

Similarly to the study of the stability, we first transform the differential equation to its equivalent system of first order different equations, and then we can apply any numerical method, like Euler's Method introduced in Module 2 to simulate its solutions.

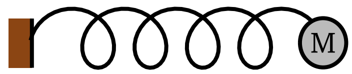
Practice Problems

1. Explain what you need to do in two different ways.
   (a) Way 1
   (b) Way 2
2. A second question.
3. A third question.

Solutions for Module 10

1. (a) Way 1 is good
   (b) Way 2 is better
2. The answer to the second question.
3.

63 Consider a spring with a mass attached to the end.



Let $x(t) =$ displacement to the right of the spring from equilibrium at time t.

Recall from Physics the following laws:

- (HL) Hooke's Law: For an elastic spring, force is proportional to negative the displacement from equilibrium.
- (NL) Newton's Second Law: Force is proportional to acceleration (the proportionality constant is called mass).
- (ML) Laws of Motion: Velocity is the time derivative of displacement and acceleration is the time derivative of velocity.

63.1 Model $x(t)$ with a differential equation.

For the remaining parts, assume the elasticity of the spring is $k = 1$ and the mass is 1.

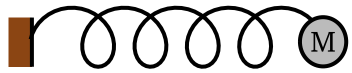
63.2 Suppose the spring is stretched 0.5m from equilibrium and then let go (at time $t = 0$).
(a) At $t = 0$, what are x, x', and x''?

(b) Modify Euler's method to approximate a solution to the initial value problem.

63.3 Introduce the auxiliary equation $y = x'$. Can the second-order spring equation be rewritten as a first-order system involving x' and y'? If so, do it.

63.4 Simulate the *system* you found in the previous part using Euler's method.

64 Recall a spring with a mass attached to the end.



$x(t)$ = displacement to the right of the spring from equilibrium at time t

We have two competing models

$$x'' = -kx \tag{N}$$

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -k & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{O}$$

where $y = x'$

XXX CHANGE EQUATION LABELS TO (A) AND (B) IN THE BOOK VERSION

64.1 Make a phase portrait for system (O). What are the axes on the phase portrait? What do you expect general solutions to look like?

64.2 Use eigenvalues/eigenvectors to find a general solution to (O). (You may use a computer to compute eigenvalues/vectors.)

64.3 Use your solution to (O) to find a general solution to (N).

65 Consider the second-order differential equation

$$x'' = -(1+x)\cdot x' + x^2 - x$$

65.1 Rewrite the second-order differential equation as a system of first-order differential equations. (Hint: you may need to introduce an auxiliary equation.)

65.2 The following Desmos link plots a phase portrait and draws an Euler approximation on the phase portrait:

`https://www.desmos.com/calculator/fvqxqp6eds`

Use the link to make a phase portrait for your system and answer the following questions:

(a) Are there initial conditions with $x(0) < 0$ so that a solution $x(t)$ is always increasing?

(b) Are there initial conditions with $x(0) < 0$ so that a solution $x(t)$ first decreases and then increases?

65.3 Show that $x(t) = 0$ is an equilibrium solution for this equation.

65.4 Use linearization and eigenvalues to classify the equilibrium $(x, x') = (0, 0)$ in phase space.

65.5 Let $x(t)$ be a solution to the original equation and suppose $x(0) = \delta_1 \approx 0$.

(a) If $x'(0) = \delta_2 \approx 0$, speculate on the long term behaviour of $x(t)$.

(b) If we put no conditions on $x'(0)$ will your answer be the same? Explain.

Introduction to Boundary Value Problems

In this module you will learn

- What is a boundary value problem
- How to solve a boundary value problem
- How to approximate the solutions of a boundary value problem

What is a boundary value problem?

Until now, we always studied differential equations with some initial conditions: conditions given all for the same value of the dependent variable. These are called *initial value problems*.

Boundary value problems are composed by one or more differential equation and a set of conditions given at different values of the dependent variable.

Example. Consider the differential equation $u''(x) = -u(x)$ with the conditions below. Which of these are initial value problems and which are boundary value problems?

1. $u(0) = 0, u'(0) = -1$ is an initial value problem, because all the conditions are given for the same value of x, namely $x = 0$.
2. $u'(100) = -2, u''(100) = 3$ is also an initial value problem, with the conditions given for the value $x = 100$.
3. $u(1) = 0, u'(2) = 0$ is a boundary value problem, because the conditions are given for different values of x: one condition at $x = 1$ and the other at $x = 2$.

Solving a boundary value problem

Typically a boundary value problem is higher order, since for first order diffferential equations we only need one condition.

So the first step is usually to transform it into a system of differential equations. Then we can solve it using the methods outlined in previous modules.

Once we have a general solution, we find the arbitrary constants using the conditions. It is exactly the same process as the one to solve an initial value problem.

Existence and Uniqueness of solution

The Theorem introduced in Module 6 doesn't apply to this type of conditions. In fact, there is no general Theorem of Existence and Uniqueness of solutions for boundary value problems.

Simulating solutions using the shooting method

Euler's method requires a full set of initial conditions to start and simulate the solution of a differential equation, so it doesn't apply to a boundary value problem.

Instead, we need to modify it into what is known as a *shooting method*.

The idea of the *shooting method* is the following:

1. Start with the boundary condition for the smallest value of $x = x_0$.
2. Introduce parameters $\vec{\xi}$ for the remaining componentsnecessary to complete an initial condition at $x = x_0$.
3. Start with some value $\vec{\xi}_0$, e.g. $\vec{\xi}_0 = \vec{0}$ and use Euler's method to simulate the solution until it reaches the other coundary condition at x_1.
4. Adjust the value of the parameter $\vec{\xi}$ and simulate again.
5. Repeat until you find a solution that satisfies the boundary condition at x_1 or comes very close to it.

Example. Simulate the solution of the boundary value problem

$$\begin{cases} u'' = -2u' - 2u \\ u(0) = 1 \\ u(1) = 1 \end{cases}$$

We first write this differential equation as a system

$$\vec{r}\,' = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} \vec{r}.$$

Our boundary condition at $x = 0$, gives us the first component of an initial condition. To be able to simulate the solution, we need to introduce a parameter for the second component of the vector $\vec{r}$, so we write:

$$\vec{r}(0) = \begin{bmatrix} 1 \\ \xi \end{bmatrix}$$

Let us start with $x_i = 0$. We can use Euler's method to simulate the solution of the system with $h = 0.01$ and obtain

$$\vec{r}(1) \approx \begin{bmatrix} 0.507 \\ -0.623 \end{bmatrix}$$

This means that the approximation of the solution at $x = 1$ is $u(1) \approx 0.507$.

Observe that we really only care about the first component of the vector $\vec{r}(x)$, which is the solution $u(x)$ of the original differential equation.

Here is a table of the successive applications of the Euler's Method with different initial conditions and how close the solution is to the boundary condition at $x = 1$:

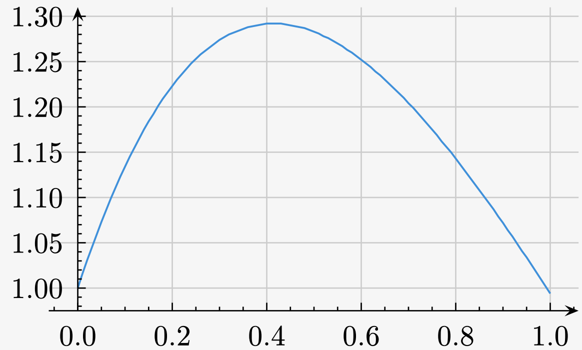
ξ	$u(1)$
0	0.507
1	0.818
2	1.130
1.5	0.974
1.75	1.052
1.625	1.013
1.5625	0.994

To adjust the value of ξ, we started with an initial guess of $\xi = 0$ and increased it until we obtained two values of $u(1)$ that are on either side of the boundary condition $u(1) = 1$. In this case, we found that $u(1) < 1$ for $\xi = 1$ and $u(1) > 1$ for $\xi = 2$. That means that the value of ξ we are looking for is between 1 and 2, so we successively bisected the interval $[1, 2]$ to find a better approximation of ξ.

There are lots of different methods to adjust the value of ξ to get closer to the boundary condition at $x = 1$. Here we just used a simple bisection method, but you can use any method you like.

We can see that the value of $u(1)$ is getting closer to the boundary condition $u(1) = 1$ as we adjust the value of ξ.

Once we are happy with the approximation, we can sketch the solution we found. Below is the grpah for the simulation with $\xi = 1.5625$.



Practice Problems

1. Explain what you need to do in two different ways.
   (a) Way 1
   (b) Way 2
2. A second question.
3. A third question.

Solutions for Module 11

1. (a) Way 1 is good
   (b) Way 2 is better
2. The answer to the second question.
3.

66 Recall the spring-mass system modeled by

$$x'' = -x$$

We would like to use the spring-mass system to ring a bell at regular intervals, so we put a hammer at the end of the spring. Whenever the displacement is maximal, the hammer strikes a bell producing a ring.

66.1 Convert the spring-mass system into a system of differential equations. Make a phase portrait for the system using the following Desmos link:

`https://www.desmos.com/calculator/fvqxqp6eds`

66.2 In the *Options Euler* on Desmos, adjust Δ and the number of steps so that simulated solutions are only shown for $t \in [0, 1]$.

Use simulations to answer the remaining questions.

66.3 You start by displacing the hammer by 1m and letting go. Is it possible that the bell rings every 1 second?

66.4 You start by displacing the hammer by 1m and giving the hammer a push. Is it possible that the bell rings every 1 second?

66.5 What is the smallest amount of time between consecutive rings (given a positive displacement)?

67 Recall the spring-mass system modeled by

$$x'' = -x$$

We would like to use the spring-mass system to ring a bell at regular intervals, so we put a hammer at the end of the spring. Whenever the displacement is maximal, the hammer strikes a bell producing a ring.

Consider the subspaces

$$S_1 = \text{span}\{\sin(t), \cos(t)\} \quad S_2 = \{A\cos(t+d) : A, d \in \mathbb{R}\}$$

67.1 What dimension is each subspace?

67.2 Which subspaces are sets of solutions to the spring-mass system?

67.3 Use what you know about complete solutions and linear algebra to prove $S_1 = S_2$.

Use your knowledge about S_1 and S_2 to analytically answer the remaining questions.

67.4 You start by displacing the hammer by 1m and letting go. Is it possible that the bell rings every 1 second?

67.5 You start by displacing the hammer by 1m and giving the hammer a push. Is it possible that the bell rings every 1 second?

67.6 What is the smallest amount of time between consecutive rings (given a positive displacement)?

68 A boundary value problem is a differential equation paired with two conditions at different values of t.

Consider the following boundary value problems:

(i)	(ii)	(iii)
$x'' = -x$	$x'' = -x$	$x'' = -x$
$x(0) = 1$	$x(0) = 1$	$x(0) = 1$
$x(\pi) = 1$	$x(\pi) = -1$	$x(\frac{\pi}{2}) = 0$

68.1 Using phase portraits and simulations, determine how many solutions each boundary value problem has.

68.2 Can you find analytic arguments to justify your conclusions?

Introduction to the Theory of Ordinary Differential Equations

This appendix is adapted from `Jiri Lebl's book "Notes on Diffy Qs"`.

We wish to ask two fundamental questions about the problem

$$y' = f(x, y), \quad y(x_0) = y_0.$$

- Does a solution *exist*?
- Is the solution *unique* (if it exists)?

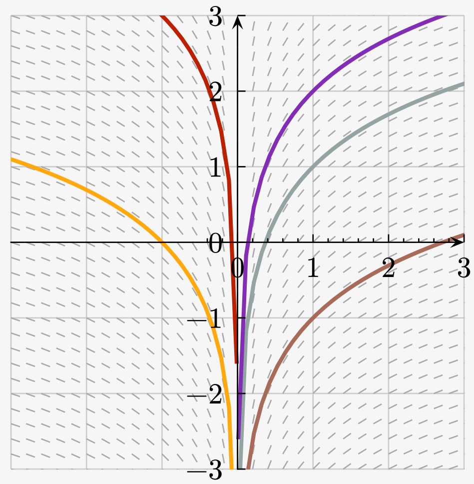
What do you think is the answer? The answer seems to be yes to both, does it not? Well, pretty much. But there are cases when the answer to either question can be no.

Since generally the equations we encounter in applications come from real life situations, it seems logical that a solution always exists. It also has to be unique if we believe our universe is deterministic. If the solution does not exist, or if it is not unique, we have probably not devised the correct model. Hence, it is good to know when things go wrong and why.

Example. Attempt to solve:

$$y' = \frac{1}{x}, \quad y(0) = 0.$$

Integrate to find the general solution $y = \ln|x| + C$. The solution does not exist at $x = 0$. See the figure below with teh slope field for this differential equation and some solutions.



Observe how the slopes get more and more vertical as x approaches 0.

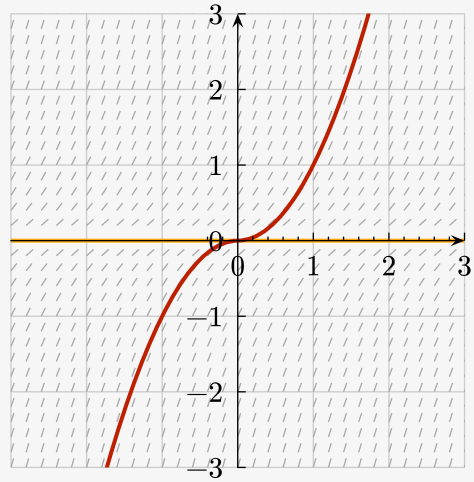
Moreover, the equation may have been written as the seemingly harmless $xy' = 1$.

Example. Solve:

$$y' = 2\sqrt{|y|}, \quad y(0) = 0.$$

See the figure below with the slope field for this differential equation and some solutions.

Module 12



Note that $y = 0$ is a solution. But another solution is the function

$$y(x) = \begin{cases} x^2 & \text{if } \ x \geq 0 \\ -x^2 & \text{if } \ x < 0 \end{cases}$$

It is hard to tell by staring at the slope field that the solution is not unique. Is there any hope? Of course there is. We have the following theorem, known as Picard's theorem

Theorem (Picard's theorem on existence and uniqueness)

If $f(x, y)$ is continuous (as a function of two variables) and $\frac{\partial f}{\partial y}$ exists and is continuous near some (x_0, y_0), then a solution to

$$y' = f(x, y), \quad y(x_0) = y_0,$$

exists (at least for x in some small interval) and is unique.

Note that the problems $y' = \frac{1}{x}$, $y(0) = 0$ and $y' = 2\sqrt{|y|}$, $y(0) = 0$ do not satisfy the hypothesis of the theorem. Even if we can use the theorem, we ought to be careful about this existence business. It is quite possible that the solution only exists for a short while.

Example. For some constant A, solve:

$$y' = y^2, \quad y(0) = A.$$

We know how to solve this equation. First assume that $A \neq 0$, so y is not equal to zero at least for some x near 0. So $x' = \frac{1}{y^2}$, so $x = -\frac{1}{y} + C$, so $y = \frac{1}{C-x}$. If $y(0) = A$, then $C = \frac{1}{A}$ so

$$y = \frac{1}{\frac{1}{A} - x}.$$

If $A = 0$, then $y = 0$ is a solution.

For example, when $A = 1$ the solution "blows up" at $x = 1$. Hence, the solution does not exist for all x even if the equation is nice everywhere. The equation $y' = y^2$ certainly looks nice.

For most of this course we will be interested in equations where existence and uniqueness holds, and in fact holds "globally" unlike for the equation $y' = y^2$.

Practice Problems

1. Sketch slope field for $y' = e^{x-y}$. How do the solutions behave as x grows? Can you guess a particular solution by looking at the slope field?
2. Sketch slope field for $y' = x^2$.
3. Sketch slope field for $y' = y^2$.
4. Is it possible to solve the equation $y' = \frac{xy}{\cos x}$ for $y(0) = 1$? Justify.
5. Is it possible to solve the equation $y' = y\sqrt{|x|}$ for $y(0) = 0$? Is the solution unique? Justify.
6. Match equations $y' = 1 - x$, $y' = x - 2y$, $y' = x(1-y)$ to slope fields. Justify.

   XXXX ADD SLOPE FIELDS
7. (Challenging) Take $y' = f(x,y)$, $y(0) = 0$, where $f(x,y) > 1$ for all x and y. If the solution exists for all x, can you say what happens to $y(x)$ as x goes to positive infinity? Explain.
8. (Challenging) Take $(y-x)y' = 0$, $y(0) = 0$.
   (a) Find two distinct solutions.
   (b) Explain why this does not violate Picard's theorem.
9. Suppose $y' = f(x,y)$. What will the slope field look like, explain and sketch an example, if you know the following about $f(x,y)$:
   (a) f does not depend on y.
   (b) f does not depend on x.
   (c) $f(t,t) = 0$ for any number t.
   (d) $f(x,0) = 0$ and $f(x,1) = 1$ for all x.
10. Find a solution to $y' = |y|$, $y(0) = 0$. Does Picard's theorem apply?
11. Take an equation $y' = (y-2x)g(x,y) + 2$ for some function $g(x,y)$. Can you solve the problem for the initial condition $y(0) = 0$, and if so what is the solution?
12. Suppose $y' = f(x,y)$ is such that $f(x,1) = 0$ for every x, f is continuous and $\frac{\partial f}{\partial y}$ exists and is continuous for every x and y.
    (a) Guess a solution given the initial condition $y(0) = 1$.
    (b) Can graphs of two solutions of the equation for different initial conditions ever intersect?
    (c) Given $y(0) = 0$, what can you say about the solution. In particular, can $y(x) > 1$ for any x? Can $y(x) = 1$ for any x? Why or why not?
13. Sketch the slope field of $y' = y^3$. Can you visually find the solution that satisfies $y(0) = 0$?
14. Is it possible to solve $y' = xy$ for $y(0) = 0$? Is the solution unique?
15. Is it possible to solve $y' = \frac{x}{x^2-1}$ for $y(1) = 0$?
16.
17. Match equations $y' = \sin x$, $y' = \cos y$, $y' = y\cos(x)$ to slope fields. Justify. XXX ADD SLOPE FIELDS
18. Suppose

$$f(y) = \begin{cases} 0 & \text{if } y > 0 \\ 1 & \text{if } y \leq 0 \end{cases}$$

    Does $y' = f(y)$, $y(0) = 0$ have a continuously differentiable solution? Does Picard apply? Why, or why not?
19. Consider an equation of the form $y' = f(x)$ for some continuous function f, and an initial condition $y(x_0) = y_0$. Does a solution exist for all x? Why or why not?

Solutions for Module 12

1.
2.
3.
4.
5.
6.
7.
8.
9.
10.
11.
12.
13.
14. Yes a solution exists. The equation is $y' = f(x, y)$ where $f(x, y) = xy$. The function $f(x, y)$ is continuous and $\frac{\partial f}{\partial y} = x$, which is also continuous near $(0, 0)$. So a solution exists and is unique. (In fact, $y = 0$ is the solution.)
15. No, the equation is not defined at $(x, y) = (1, 0)$.
16. The answer to the second question.
17. (a) $y' = \cos y$
    (b) $y' = y \cos(x)$
    (c) $y' = \sin x$.

    Justification left to reader.
18. Picard does not apply as f is not continuous at $y = 0$. The equation does not have a continuously differentiable solution. Suppose it did. Notice that $y'(0) = 1$. By the first derivative test, $y(x) > 0$ for small positive x. But then for those x, we have $y'(x) = f(y(x)) = 0$. It is not possible for y' to be continous, $y'(0) = 1$ and $y'(x) = 0$ for arbitrarily small positive x.
19. The solution is $y(x) = \int_{x_0}^{x} f(s) ds + y_0$, and this does indeed exist for every x.

69 Whether a solution to a differential equation exists or is unique is a *hard* question with many partial answers.

Recall the Theorem:

Theorem (Existence and Uniqueness II)

Let $F(t, x, x') = 0$ with $x(t_0) = x_0$ describe an initial value problem.

- IF $F(t, x, x') = x'(t) + p(t)x(t) + g(t)$ for some functions p and g
- AND p and g are continuous on an open interval I containing t_0
- THEN the initial value problem has a unique solution on I.

69.1 The theorem expresses differential equations in the form $F(t, x, x', x'', \ldots) = 0$ (i.e. as a level set of some function F).

Rewrite the following differential equations in the form $F(t, x, x', x'', \ldots) = 0$:

(a) $x'' = -kx$

(b) $x'' = -x \cdot x' + x^2$

(c) $x''' = (x')^2 - \cos x$

69.2 Which of the following does the theorem say *must* have a unique solution on an interval containing 0?

(a) $y' = \frac{3}{2}y^{\frac{1}{3}}$ with $y(0) = 0$

(b) $x'(t) = \lfloor t \rfloor x(t)$ with $x(0) = 0$

(c) $x'(t) = \lfloor t - \frac{1}{2} \rfloor x(t) + t^2$ with $x(0) = 0$

Note: $\lfloor x \rfloor$ is the *floor* of x, i.e., the largest integer less than or equal to x.

Spreadsheets and Programming

In this appendix you will learn:

- The basics of spreadsheet usage
- How to set up formulas in spreadsheets

Spreadsheets serve as a powerful data analysis and management tool. Commonly associated with Microsoft Excel, spreadsheets provide an interactive table that, for us, is ideal for implementing some of the basic algorithms for simulating solutions to differential equations.

Spreadsheet Basics

The interface to a spreadsheet is a table of editable *cells*. Every column of the spreadsheet is indexed by a capital letter (*A*, *B*, *C*, …, *Z*, *AA*, *AB*, …, with the column names turning to double letters after *Z*), and the rows indexed by numbers, starting at *1*.

	A	B	C	D	E
1					
2		*Cell "B2"*			*Cell "E2"*
3					
4	*Cell "A4"*				
5					

Text or numbers can be entered into any cell of a spreadsheet.

Formulas

If a cell's contents start with an *equals sign* `=`, what follows will be interpreted as a formula. Formulas can include math expressions (`+`, `-`, `*`, `/`) as well as references to other cells.

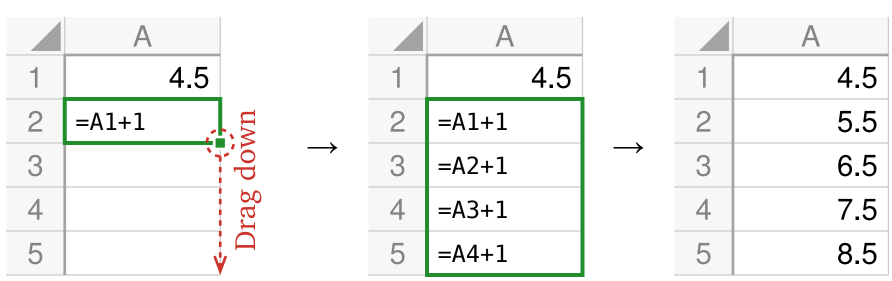
For example, when the following spreadsheet is evaluated, the contents of `B2` will be `9` and the contents of `C2` will be `10`.

	A	B	C
1	4.5		
2		=A1*2	=B2+1
3			
4			
5			

→

	A	B	C
1	4.5		
2		9	10
3			
4			
5			

Formulas can be *interpolated* when copy/pasted or extended to other cells. In the following example, the formula `=A1+1` is being interpolated to "add one to the previous cell".

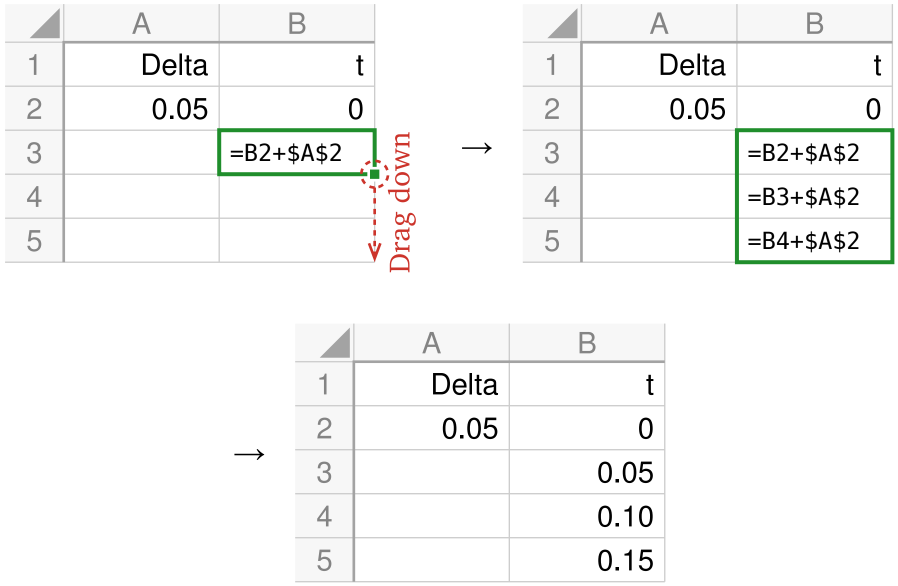


Preventing Interpolation

What if you don't want interpolation? Or you only want it in particular places? For example, instead of adding `1` to the previous cell, let us add the value of `A2` to the previous cell.

We can prevent interpolation of a row/column/both by adding a "$" before the row/column in a cell index. For example,

- `=A$1` would interpolate the column but not the row,
- `=$A1` would interpolate the row but not the column, and
- `=$A$1` would not interpolate either.



Pitfalls of Formulas

XXX Finish (some stuff about order of operations, ### showing up in a cell, etc.)

Graphing

XXX Finish

Practice Problems

1. Explain what you need to do in two different ways.
   (a) Way 1
   (b) Way 2
2. A second question.
3. A third question.

Solutions for Appendix A

1. (a) Way 1 is good
   (b) Way 2 is better
2. The answer to the second question.
3.

Separable Equations

This appendix is adapted from `Jiri Lebl's book "Notes on Diffy Qs"`.

When a differential equation is of the form $y' = f(x)$, we integrate: $y = \int f(x)dx + C$. Unfortunately, simply integrating no longer works for the general form of the equation $y' = f(x, y)$. Integrating both sides yields the rather unhelpful expression

$$y(x) = \int f(x, y(x))dx + C.$$

Notice the dependence on y in the integral.

Separable equations

We say a differential equation is **separable** if we can write it as

$$y' = f(x)g(y),$$

for some functions $f(x)$ and $g(y)$. Let us write the equation in the Leibniz notation

$$\frac{dy}{dx} = f(x)g(y).$$

Then we rewrite the equation as

$$\frac{dy}{g(y)} = f(x)dx.$$

Both sides look like something we can integrate. We obtain

$$\int \frac{dy}{g(y)} = \int f(x)dx + C.$$

If we can find closed form expressions for these two integrals, we can, perhaps, solve for y.

Example. Take the equation

$$y' = xy.$$

Find the general solution.

Note that $y = 0$ is a solution. We will remember that fact and assume $y \neq 0$ from now on, so that we can divide by y. Write the equation as $\frac{dy}{dx} = xy$ or $\frac{dy}{y} = xdx$. Then

$$\int \frac{dy}{y} = \int xdx + C.$$

We compute the antiderivatives to get

$$\ln|y| = \frac{x^2}{2} + C,$$

or

$$|y| = e^{\frac{x^2}{2}+C} = e^{\frac{x^2}{2}}e^C = De^{\frac{x^2}{2}},$$

where $D > 0$ is some constant. Because $y = 0$ is also a solution and because of the absolute value, we can write:

$$y = De^{\frac{x^2}{2}},$$

for any number D (including zero or negative).

We check:

$$y' = Dxe^{\frac{x^2}{2}} = x\left(De^{\frac{x^2}{2}}\right) = xy.$$

Yay!

You may be worried that we integrated in two different variables. We seemingly did a different operation to each side. Perhaps we should be a little bit more careful and work through this method more rigorously. Consider

$$\frac{dy}{dx} = f(x)g(y).$$

We rewrite the equation as follows. Note that $y = y(x)$ is a function of x and so is $\frac{dy}{dx}$!

$$\frac{1}{g(y)}\frac{dy}{dx} = f(x).$$

We integrate both sides with respect to x:

$$\int \frac{1}{g(y)}\frac{dy}{dx}dx = \int f(x)dx + C.$$

We use the change of variables formula (substitution) on the left hand side:

$$\int \frac{1}{g(y)}dy = \int f(x)dx + C.$$

And we are done.

Implicit solutions

We sometimes get stuck even if we can do the integration. Consider the separable equation

$$y' = \frac{xy}{y^2+1}.$$

We separate variables,

$$\left(\frac{y^2+1}{y}\right)dy = \left(y + \frac{1}{y}\right)dy = xdx.$$

We integrate to get

$$\frac{y^2}{2} + \ln|y| = \frac{x^2}{2} + C,$$

or perhaps the less intimidating expression (where $D = 2C$)

$$y^2 + 2\ln|y| = x^2 + D.$$

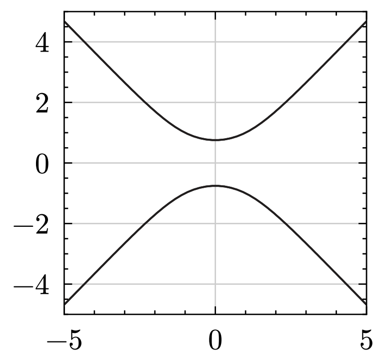
It is not easy to find the solution explicitly—it is hard to solve for y. We, therefore, leave the solution in this form and call it an *implicit solution*. It is still easy to check that an implicit solution satisfies the differential equation. In this case, we differentiate with respect to x, and remember that y is a function of x, to get

$$y'\left(2y+\frac{2}{y}\right)=2x.$$

Multiply both sides by y and divide by $2(y^2+1)$ and you will get exactly the differential equation. We leave this computation to the reader.

If you have an implicit solution, and you want to compute values for y, you might have to be tricky. You might get multiple solutions y for each x, so you have to pick one. Sometimes you can graph x as a function of y, and then turn your paper to see a graph. Sometimes you have to do more.

Computers are also good at some of these tricks. More advanced mathematical software usually has some way of plotting solutions to implicit equations. For example, for $D=0$, if you plot all the points (x,y) that are solutions to $y^2+2\ln|y|=x^2$, you find the two curves in the figure below.



This is not quite a graph of a function. For each x there are two choices of y. To find a function, you have to pick one of these two curves. You pick the one that satisfies your initial condition if you have one. For instance, the top curve satisfies the condition $y(1)=1$. So for each D, we really got two solutions. As you can see, computing values from an implicit solution can be somewhat tricky. But sometimes, an implicit solution is the best we can do.

The equation above also has the solution $y=0$. So the general solution is

$$y^2+2\ln|y|=x^2+D, \quad \text{and} \quad y=0.$$

Sometimes these extra solutions that came up due to division by zero such as $y=0$ are called *singular solutions*.

Examples of separable equations

Example.

Solve $x^2y'=1-x^2+y^2-x^2y^2$, $y(1)=0$.

Factor the right-hand side

$$x^2y'=(1-x^2)(1+y^2).$$

Separate variables, integrate, and solve for y:

$$
\begin{aligned}
\frac{y'}{1+y^2} &= \frac{1-x^2}{x^2}, \\
\frac{y'}{1+y^2} &= \frac{1}{x^2} - 1, \\
\arctan(y) &= -\frac{1}{x} - x + C, \\
y &= \tan\left(-\frac{1}{x} - x + C\right)
\end{aligned}
$$

Solve for the initial condition, $0 = \tan(-2 + C)$ to get $C = 2$ (or $C = 2 + \pi$, or $C = 2 + 2\pi$, etc.). The particular solution we seek is, therefore,

$$y = \tan\left(\left(-\frac{1}{x}\right) - x + 2\right).$$

Example.

Bob made a cup of coffee, and Bob likes to drink coffee only once reaches 60 degrees Celsius and will not burn him. Initially at time $t = 0$ minutes, Bob measured the temperature and the coffee was 89 degrees Celsius. One minute later, Bob measured the coffee again and it had 85 degrees. The temperature of the room (the ambient temperature) is 22 degrees. When should Bob start drinking?

Let T be the temperature of the coffee in degrees Celsius, and let A be the ambient (room) temperature, also in degrees Celsius. Newton's law of cooling states that the rate at which the temperature of the coffee is changing is proportional to the difference between the ambient temperature and the temperature of the coffee. That is,

$$\frac{dT}{dt} = k(A - T),$$

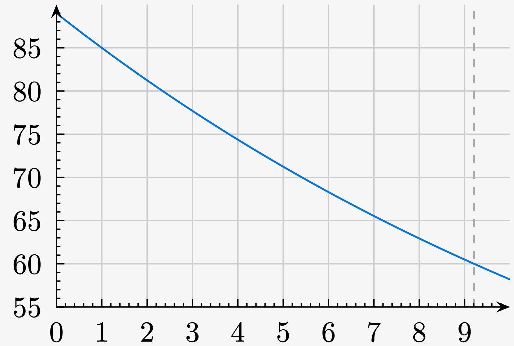
for some positive constant k. For our setup $A = 22$, $T(0) = 89$, $T(1) = 85$. We separate variables and integrate (let C and D denote arbitrary constants):

$$
\begin{aligned}
\left(\frac{1}{T-A}\right)\frac{dT}{dt} &= -k, \\
\ln(T - A) &= -kt + C, \quad \text{note that} \quad T - A > 0, \\
T - A &= De^{-kt}, \\
T &= A + De^{-kt}
\end{aligned}
$$

That is, $T = 22 + De^{-kt}$. We plug in the first condition: $89 = T(0) = 22 + D$, and hence $D = 67$. So $T = 22 + 67e^{-kt}$. The second condition says $85 = T(1) = 22 + 67e^{-k}$. Solving for k, we get $k = -\ln\left(\frac{85-22}{67}\right) \approx 0.0616$. Now we solve for the time t that gives us a temperature of 60 degrees. Namely, we solve

$$60 = 22 + 67e^{-0.0616t}$$

to get $t = -\frac{\ln\left(\frac{60-22}{67}\right)}{0.0616} \approx 9.21$ minutes. So Bob can begin to drink the coffee at just over 9 minutes from the time Bob made it. That is probably about the amount of time it took us to calculate how long it would take.



Example. Find the general solution to $y' = \frac{-xy^2}{3}$ (including any singular solutions).

First note that $y = 0$ is a solution (a singular solution). Now assume that $y \neg = 0$.

$$-\frac{3}{y^2}y' = x,$$

$$\frac{3}{y} = \frac{x^2}{2} + C,$$

$$y = \frac{3}{\frac{x^2}{2} + C}$$

$$= \frac{6}{x^2 + 2C}$$

So the general solution is

$$y = \frac{6}{x^2 + 2C} \quad \text{and} \quad y = 0.$$

Practice Problems

1. Solve $y' = \frac{x}{y}$.
2. Solve $y' = x^2y$.
3. Solve $\frac{dx}{dt} = (x^2 - 1)t$, for $x(0) = 0$.
4. Solve $\frac{dx}{dt} = x\sin(t)$, for $x(0) = 1$.
5. Solve $\frac{dy}{dx} = xy + x + y + 1$. Hint: Factor the right-hand side.
6. Solve $xy' = y + 2x^2y$, where $y(1) = 1$.
7. Solve $\frac{dy}{dx} = \frac{y^2+1}{x^2+1}$, for $y(0) = 1$.
8. Find an implicit solution for $\frac{dy}{dx} = \frac{x^2+1}{y^2+1}$, for $y(0) = 1$.
9. Find an explicit solution for $y' = xe^{-y}$, $y(0) = 1$.
10. Find an explicit solution for $xy' = e^{-y}$, for $y(1) = 1$.
11. Find an explicit solution for $y' = ye^{-x^2}$, $y(0) = 1$. It is alright to leave a definite integral in your answer.
12. Suppose a cup of coffee is at 100 degrees Celsius at time $t = 0$, it is at 70 degrees at $t = 10$ minutes, and it is at 50 degrees at $t = 20$ minutes. Compute the ambient temperature.
13. Solve $y' = 2xy$.
14. Solve $x' = 3xt^2 - 3t^2$, $x(0) = 2$.
15. Find an implicit solution for $x' = \frac{1}{3x^2+1}$, $x(0) = 1$.
16. Find an explicit solution to $xy' = y^2$, $y(1) = 1$.
17. Find an implicit solution to $y' = \frac{\sin(x)}{\cos(y)}$.
18. Take the coffee example from page 150 with the same numbers: 89 degrees at $t = 0$, 85 degrees at $t = 1$, and ambient temperature of 22 degrees. Suppose these temperatures were measured with precision of ± 0.5 degrees. Given this imprecision, the time it takes the coffee to cool to (exactly) 60 degrees is also only known in a certain range. Find this range. Hint: Think about what kind of

error makes the cooling time longer and what shorter.

19. A population x of rabbits on an island is modeled by $x' = x - \left(\frac{1}{1000}\right)x^2$, where the independent variable is time in months. At time $t = 0$, there are 40 rabbits on the island.

    (a) Find the solution to the equation with the initial condition.

    (b) How many rabbits are on the island in 1 month, 5 months, 10 months, 15 months (round to the nearest integer).

Solutions for Appendix B

1.
2.
3.
4.
5.
6.
7.
8.
9.
10.
11.
12.
13. The solution is $y = Ce^{x^2}$.
14. The solution is $x = e^{t^3} + 1$.
15. The solution is $x^3 + x = t + 2$.
16. The solution is $y = \frac{1}{1-\ln(x)}$.
17. The solution is $\sin(y) = -\cos(x) + C$.
18. The range is approximately 7.45 to 12.15 minutes.
19. (a) $x = \frac{1000e^t}{e^t+24}$.

    (b) 102 rabbits after one month, 861 after 5 months, 999 after 10 months, 1000 after 15 months.

Integrating Factors

This appendix is adapted from `Jiri Lebl's book "Notes on Diffy Qs"`.

One of the most important types of equations we will learn to solve are the so-called **linear equations**. In fact, the majority of the course is about linear equations. In this section we focus on the **first order linear equation**. A first order equation is linear if we can put it into the form:

$$y' + p(x)y = f(x).$$

The word "linear" means linear in y and y'; no higher powers nor functions of y or y' appear. The dependence on x can be more complicated.

Solutions of linear equations have nice properties. For example, the solution exists wherever $p(x)$ and $f(x)$ are defined, and has essentially the same regularity (read: it is just as nice). But most importantly for us right now, there is a method for solving linear first order equations.

The trick is to rewrite the left-hand side of $y' + p(x)y = f(x)$ as a derivative of a product of y with another function. To this end, we find a function $r(x)$ such that

$$r(x)y' + r(x)p(x)y = \frac{d}{dx}[r(x)y].$$

This is the left-hand side of $y' + p(x)y = f(x)$ multiplied by $r(x)$. If we multiply $y' + p(x)y = f(x)$ by $r(x)$, we obtain

$$\frac{d}{dx}[r(x)y] = r(x)f(x).$$

We can now integrate both sides, which we can do as the right-hand side does not depend on y and the left-hand side is written as a derivative of a function. After the integration, we solve for y by dividing by $r(x)$. The function $r(x)$ is called the **integrating factor** and the method is called the **integrating factor method**.

We are looking for a function $r(x)$, such that if we differentiate it, we get the same function back multiplied by $p(x)$. That seems like a job for the exponential function! Let

$$r(x) = e^{\int p(x)dx}.$$

We compute:

$$\begin{aligned} y' + p(x)y &= f(x), \\ e^{\int p(x)dx}y' + e^{\int p(x)dx}p(x)y &= e^{\int p(x)dx}f(x), \\ \frac{d}{dx}\left[e^{\int p(x)dx}y\right] &= e^{\int p(x)dx}f(x), \\ e^{\int p(x)dx}y &= \int e^{\int p(x)dx}f(x)dx + C, \\ y &= e^{-\int p(x)dx}\left(\int e^{\int p(x)dx}f(x)dx + C\right). \end{aligned}$$

Of course, to get a closed form formula for y, we need to be able to find a closed form formula for the integrals appearing above.

Example. Solve

$$y' + 2xy = e^{x-x^2}, \quad y(0) = -1.$$

First note that $p(x) = 2x$ and $f(x) = e^{x-x^2}$. The integrating factor is $r(x) = e^{\int p(x)\,dx} = e^{x^2}$. We multiply both sides of the equation by $r(x)$ to get

$$e^{x^2}y' + 2xe^{x^2}y = e^{x-x^2}e^{x^2},$$
$$\frac{d}{dx}\left[e^{x^2}y\right] = e^x.$$

We integrate

$$e^{x^2}y = e^x + C,$$
$$y = e^{x-x^2} + Ce^{-x^2}.$$

Next, we solve for the initial condition $-1 = y(0) = 1 + C$, so $C = -2$. The solution is

$$y = e^{x-x^2} - 2e^{-x^2}.$$

Note that we do not care which antiderivative we take when computing $e^{\int p(x)dx}$. You can always add a constant of integration, but those constants will not matter in the end.

Exercise: Try it! Add a constant of integration to the integral in the integrating factor and show that the solution you get in the end is the same as what we got above.

Advice: Do not try to remember the formula for y itself, that is way too hard. It is easier to remember the process and repeat it.

Since we cannot always evaluate the integrals in closed form, it is useful to know how to write the solution in definite integral form. A definite integral is something that you can plug into a computer or a calculator. Suppose we are given

$$y' + p(x)y = f(x), \quad y(x_0) = y_0.$$

Look at the solution and write the integrals as definite integrals.

$$y(x) = e^{-\int_{x_0}^{x} p(s)\,ds}\left(\int_{x_0}^{x} e^{\int_{x_0}^{t} p(s)\,ds} f(t)\,dt + y_0\right).$$

You should be careful to properly use dummy variables here. If you now plug such a formula into a computer or a calculator, it will be happy to give you numerical answers.

Exercise: Check that $y(x_0) = y_0$ in formula above.

Exercise: Write the solution of the following problem as a definite integral, but try to simplify as far as you can. You will not be able to find the solution in closed form.

$$y' + y = e^{x^2-x}, \quad y(0) = 10.$$

Remark: Before we move on, we should note some interesting properties of linear equations. First, for the linear initial value problem $y' + p(x)y = f(x)$, $y(x_0) = y_0$, there is an explicit formula above for the solution. Second, it follows from the formula above that if $p(x)$ and $f(x)$ are continuous on some interval (a, b), then the solution $y(x)$ exists and is differentiable on (a, b). Compare with the simple nonlinear example we have seen previously, $y' = y^2$, and compare to the Picard-Lindelöf Theorem (`https://en.wikipedia.org/wiki/Picard-Lindelof_theorem`).

Example. Let us discuss a common simple application of linear equations. Real life applications of this type of problem include figuring out the concentration of chemicals in bodies of water (rivers and lakes).

XXX

A 100 litre tank contains 10 kilograms of salt dissolved in 60 litres of water. Solution of water and salt (brine) with concentration of 0.1 kilograms per litre is flowing in at the rate of 5 litres a minute. The solution in the tank is well stirred and flows out at a rate of 3 litres a minute. How much salt is in the tank when the tank is full?

Let us come up with the equation. Let x denote the kilograms of salt in the tank, let t denote the time in minutes. For a small change Δt in time, the change in x (denoted Δx) is approximately

$$\Delta x \approx (\text{rate in} \cdot \text{concentration in})\Delta t - (\text{rate out} \cdot \text{concentration out})\Delta t.$$

Dividing through by Δt and taking the limit $\Delta t \to 0$, we see that

$$\frac{dx}{dt} = (\text{rate in} \cdot \text{concentration in}) - (\text{rate out} \cdot \text{concentration out}).$$

In our example,

$$\begin{aligned} \text{rate in} &= 5, \\ \text{concentration in} &= 0.1, \\ \text{rate out} &= 3, \\ \text{concentration out} &= \frac{x}{\text{volume}} = \frac{x}{60 + (5-3)t}. \end{aligned}$$

Our equation is, therefore,

$$\frac{dx}{dt} = (5 \cdot 0.1) - \left(3 \frac{x}{60 + 2t}\right).$$

Or in the form $y' + p(x)y = f(x)$,

$$\frac{dx}{dt} + \left(\frac{3}{60 + 2t}\right) x = 0.5.$$

Let us solve. The integrating factor is

$$r(t) = \exp\left(\int \frac{3}{60 + 2t} dt\right) = \exp\left(\left(\frac{3}{2}\right) \ln(60 + 2t)\right) = (60 + 2t)^{\frac{3}{2}}.$$

We multiply both sides of the equation to get

$$\begin{aligned} (60 + 2t)^{\frac{3}{2}} \frac{dx}{dt} + (60 + 2t)^{\frac{3}{2}} \left(\frac{3}{60 + 2t}\right) x &= 0.5(60 + 2t)^{\frac{3}{2}}, \\ \frac{d}{dt}\left[(60 + 2t)^{\frac{3}{2}} x\right] &= 0.5(60 + 2t)^{\frac{3}{2}}, \\ (60 + 2t)^{\frac{3}{2}} x &= \int 0.5(60 + 2t)^{\frac{3}{2}} dt + C, \end{aligned}$$

So

Appendix C — Integrating Factors

$$x = (60+2t)^{-\frac{3}{2}} \int \frac{(60+2t)^{\frac{3}{2}}}{2} dt + C(60+2t)^{-\frac{3}{2}},$$
$$x = (60+2t)^{-\frac{3}{2}} \left(\frac{1}{10}\right)(60+2t)^{\frac{5}{2}} + C(60+2t)^{-\frac{3}{2}},$$
$$x = \frac{60+2t}{10} + C(60+2t)^{-\frac{3}{2}}.$$

To find C, note that at $t = 0$, we have $x = 10$. That is,

$$10 = x(0) = \frac{60}{10} + C(60)^{-\frac{3}{2}} = 6 + C(60)^{-\frac{3}{2}},$$

or

$$C = 4\left(60^{\frac{3}{2}}\right) \approx 1859.03.$$

We know 5 litres per minute are flowing in and 3 litres per minute are flowing out, so the volume is increasing by 2 litres a minute. So the tank is full when $60 + 2t = 100$, or when $t = 20$. We are interested in the value of x when the tank is full, that is we want to compute $x(20)$:

$$x(20) = \frac{60+40}{10} + C(100)^{-\frac{3}{2}}$$
$$\approx 10 + 1859.03(100)^{-\frac{3}{2}} \approx 11.86.$$

There are 11.86 kg of salt in the tank when it is full. See the figure for the graph of x over t. XXX

The concentration when the tank is full is approximately $\frac{11.86}{100} = 0.1186$ kg/litre, and we started with $\frac{1}{6}$ or approximately 0.1667 kg/litre.

XXX Figure

In the practice problems, feel free to leave answer as a definite integral if a closed form solution cannot be found. If you can find a closed form solution, you should give that.

Practice Problems

1. Solve $y' + xy = x$.
2. Solve $y' + 6y = e^x$.
3. Solve $y' + 3x^2y = \sin(x)e^{\{-x^3\}}$, with $y(0) = 1$.
4. Solve $y' + \cos(x)y = \cos(x)$.
5. Solve $\frac{1}{x^2+1}y' + xy = 3$, with $y(0) = 0$.
6. Suppose there are two lakes located on a stream. Clean water flows into the first lake, then the water from the first lake flows into the second lake, and then water from the second lake flows further downstream. The in and out flow from each lake is 500 litres per hour. The first lake contains 100 thousand litres of water and the second lake contains 200 thousand litres of water. A truck with 500 kg of toxic substance crashes into the first lake. Assume that the water is being continually mixed perfectly by the stream.
   (a) Find the concentration of toxic substance as a function of time in both lakes.
   (b) When will the concentration in the first lake be below 0.001 kg per litre?
   (c) When will the concentration in the second lake be maximal?
7. Newton's law of cooling states that $\frac{dx}{dt} = -k(x - A)$ where x is the temperature, t is time, A is the ambient temperature, and $k > 0$ is a constant. Suppose that

$A = A_0 \cos(\omega t)$ for some constants A_0 and ω. That is, the ambient temperature oscillates (for example night and day temperatures).

(a) Find the general solution.

(b) In the long term, will the initial conditions make much of a difference? Why or why not?

8. Initially 5 grams of salt are dissolved in 20 litres of water. Brine with concentration of salt 2 grams of salt per litre is added at a rate of 3 litres a minute. The tank is mixed well and is drained at 3 litres a minute. How long does the process have to continue until there are 20 grams of salt in the tank?
9. Initially a tank contains 10 litres of pure water. Brine of unknown (but constant) concentration of salt is flowing in at 1 litre per minute. The water is mixed well and drained at 1 litre per minute. In 20 minutes there are 15 grams of salt in the tank. What is the concentration of salt in the incoming brine?
10. Solve $y' + 3x^2y = x^2$.
11. Solve $y' + 2\sin(2x)y = 2\sin(2x)$, $y(\frac{\pi}{2}) = 3$.A third question.
12. Suppose a water tank is being pumped out at 3 L/min. The water tank starts at 10 L of clean water. Water with toxic substance is flowing into the tank at 2 L/min, with concentration $20t$ g/L at time t. When the tank is half empty, how many grams of toxic substance are in the tank (assuming perfect mixing)?
13. There is bacteria on a plate and a toxic substance is being added that slows down the rate of growth of the bacteria. That is, suppose that $\frac{dP}{dt} = (2 - 0.1t)P$. If $P(0) = 1000$, find the population at $t = 5$.
14. A cylindrical water tank has water flowing in at I cubic meters per second. Let A be the area of the cross section of the tank in square meters. Suppose water is flowing out from the bottom of the tank at a rate proportional to the height of the water level. Set up the differential equation for h, the height of the water, introducing and naming constants that you need. You should also give the units for your constants.

Solutions for Appendix C

1.
2.
3.
4.
5.
6.
7.
8.
9.
10. $y = Ce^{-x^3} + \frac{1}{3}$
11. $y = 2e^{\cos(2x)+1} + 1$
12. 250 grams
13. $P(5) = 1000e^{2\times5-0.05\times5^2} = 1000e^{8.75} \approx 6.31 \times 10^6$
14. $Ah' = I - kh$, where k is a constant with units $\frac{m^2}{s}$.

Series Solutions

Practice Problems

1. Explain what you need to do in two different ways.
   (a) Way 1
   (b) Way 2
2. A second question.
3. A third question.

Solutions for Appendix D

1. (a) Way 1 is good
   (b) Way 2 is better
2. The answer to the second question.
3.

Reduction of Order

This appendix is adapted from `Jiri Lebl's book "Notes on Diffy Qs"`.

In general finding solutions to differential equations is difficult, if not impossible.

However, if we already know one solution to a differential equation, we can use that solution to simplify the differential equation and make it easier to find the other solutions. The method for doing this is called **reduction of order**.

This is an analgous procedure to finding the divisors of a number. For example if I want to find the divisors of 364 and I already know that 7 is a divisor, then I can divide 364 by 7 to get 52. Now I know that all the other divisors of 364 must also be divisors of 52, so I can focus on 52 (which is smaller and more manageable) instead of 364.

The idea is that if we somehow found y_1 as a solution of $y'' + p(x)y' + q(x)y = 0$, then we could simplify this differential to a first order equation by assuming that a second solution is of the form $y_2(x) = y_1(x)v(x)$, where $v(x)$ is some function we need to find.

Example. Find $y_2(x)$.

We just need to find v. We plug y_2 into the equation:

$$\begin{aligned} 0 = y_2'' + p(x)y_2' + q(x)y_2 &= (y_1''v + 2y_1'v' + y_1v'') + p(x)(y_1'v + y_1v') + q(x)(y_1v) \\ &= y_1v'' + (2y_1' + p(x)y_1)v' + \cancel{(y_1'' + p(x)y_1' + q(x)y_1)}v \end{aligned}$$

In other words, $y_1v'' + (2y_1' + p(x)y_1)v' = 0$. Using $w = v'$, we have the first order linear equation

$$y_1w' + (2y_1' + p(x)y_1)w = 0.$$

To find the solution $y_2(x)$, we only need to solve this differential equation, which is a first-order differential equation in w.

After solving this equation for w (using integrating factor or separation of variables), we find v by anti-differentiating w. We then form y_2 by computing y_1v.

Since we have a formula for the solution to the first order linear equation, we can write a formula for y_2:

$$y_2(x) = y_1(x)\int\left(\frac{e^{-\int(p(x)dx)}}{y_1(x)^2}dx\right).$$

Note that this formula is only valid for a linear second-order differential equation.

However, it is much easier to remember that we just need to try $y_2(x) = y_1(x)v(x)$ and find $v(x)$ as we did above. Also, the technique works for higher order equations too: You get to reduce the order for each solution you find. So it is better to remember how to do it rather than a specific formula.

Practice Problems

1. Suppose y_1 is a solution to $y'' + p(x)y' + q(x)y = 0$. By directly plugging into the equation, show that

$$y_2(x) = y_1(x)\int\left(\frac{e^{-\int(p(x)dx)}}{y_1(x)^2}dx\right)$$

is also a solution.

2. Take $2x^2y'' + xy' - 3y = 0$ for $x > 0$.

(a) Show that $y = \frac{1}{x}$ is a solution.
(b) Use reduction of order to find a second linearly independent solution.
(c) Write down the general solution.

3. (Chebyshev's equation of order 1) Take $(1-x^2)y'' - xy' + y = 0$.
   (a) Show that $y = x$ is a solution.
   (b) Use reduction of order to find a second linearly independent solution.
   (c) Write down the general solution.
4. (Hermite's equation of order 2) Take $y'' - 2xy' + 4y = 0$.
   (a) Show that $y = 1 - 2x^2$ is a solution.
   (b) Use reduction of order to find a second linearly independent solution. (It's OK to leave a definite integral in the formula.)
   (c) Write down the general solution.
5. Take $(2-x)y''' + (2x-3)y'' - xy' + y = 0$ for $x < 2$.
   (a) Show that $y = e^x$ is a solution.
   (b) Use reduction of order to reduce this differential equation to a lower order differential equation.
   (c) Write down the general solution.
   (d) Why did we need the condition $x < 2$?

Solutions for Appendix E

1.
2.
3.
4.
5.

Practice Problems

1. Explain what you need to do in two different ways.
   (a) Way 1
   (b) Way 2
2. A second question.
3. A third question.

Solutions for Appendix F

1. (a) Way 1 is good
   (b) Way 2 is better
2. The answer to the second question.
3.

Complex Numbers

In this appendix[25] you will learn:

- What a complex numbers is,
- The geometry of complex numbers, and
- The basics of manipulating complex numbers.

Preliminaries

The imaginary numbers are built from a special symbol i, whose square is $i^2 = -1$. An *imaginary* number is a pure multiple of i, like bi, where b is real. A *complex* number is a sum of a real and imaginary number, and generally looks like $a + bi$. The set of all complex numbers, denoted $\mathbb{C}$, is

$$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}.$$

To add and multiply complex numbers, you can treat "i" like a variable, use the usual rules of arithmetic, and then simplify any occurrences of i^2 to -1. For example,

$$(1 + 2i) + (3 + 4i) = (1 + 3) + (2 + 4)i = 4 + 6i$$

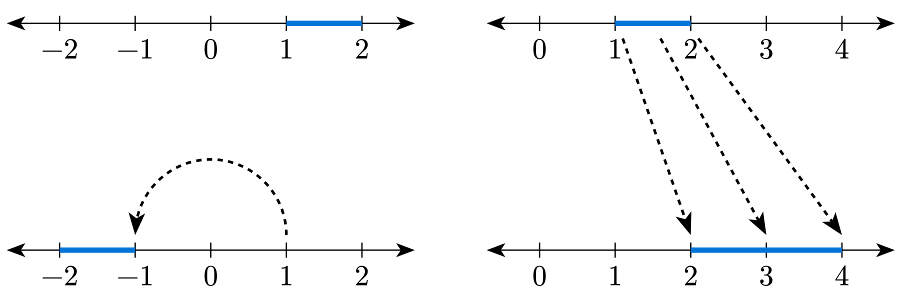
and

$$(1 + 2i) \cdot (3 + 4i) = 3 + 4i + 6i + 8i^2 = (3 - 8) + (4 + 6)i = -5 + 10i.$$

If $z = a + bi$, we call a the *real part* of z and b the *imaginary part* of z. Note: both the real part and the imaginary part of a complex number are *real numbers*.

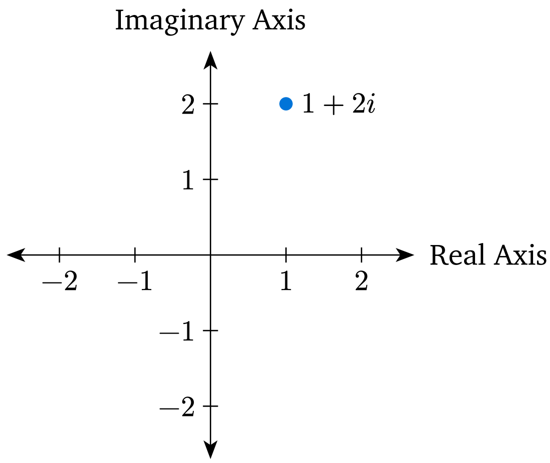
Geometry of the Complex Numbers

The real numbers ($\mathbb{R}$) can be identified with a number line. Multiplying by a real number stretches, compresses, or reflects this number line (depending on whether the number is greater than 1, less than 1, or negative). For example, consider the interval $[1, 2]$. The figure on the left shows the effect of multiplication by -1 and on the right multiplication by 2:

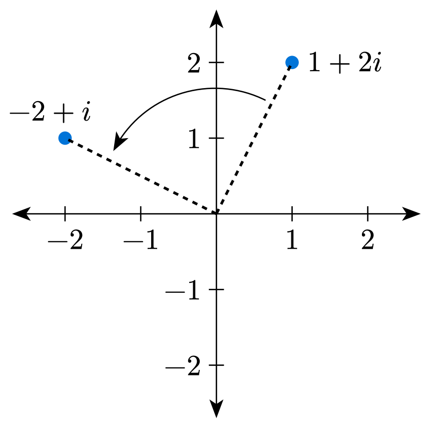


Just like the real numbers, the complex numbers have a geometric meaning. A complex number $1 + 2i$ can be thought of as a point in the complex plane (with one axis the *real axis* and the other axis the *imaginary axis*).

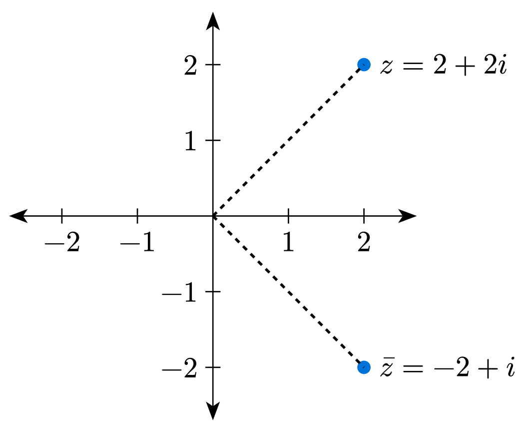
[25]Special thanks to Rhupert Ashmore-Sharpe for authoring the bulk of this appendix.



If we multiply $1+2i$ by i, we get $i \cdot (1+2i) = i + 2i^2 = -2+i$, which geometrically corresponds to a rotation counter-clockwise by 90 degrees.



For a complex number $z = a + bi$, we define the **complex conjugate** as $\bar{z} = a - bi$. Geometrically, taking the complex conjugate of a complex number reflects it over the real axis. For example, take $z = 2 + 2i$ and its complex conjugate $\bar{z} = 2 - 2i$.



If $z = a + bi$, then $z \cdot \bar{z} = a^2 + b^2$, which is a real number. The **absolute value** or **modulus** of a complex number $z = a + bi$ is

$$|z| = \sqrt{a^2 + b^2} = \sqrt{z \cdot \bar{z}}.$$

Here is a list of other nice properties of the complex conjugate (verify these by working out the algebra yourself):

- $\overline{z^n} = \bar{z}^n$
- $\overline{\frac{z_1}{z_2}} = \frac{\bar{z}_1}{\bar{z}_2}$

- $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

The reciprocal, $\frac{1}{z}$, of a complex number can be simplified by multiplying the numerator and denominator by $\bar{z}$:

$$\frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{\bar{z}}{|z|^2}$$

In this way, you can simplify a complex division problem. For example

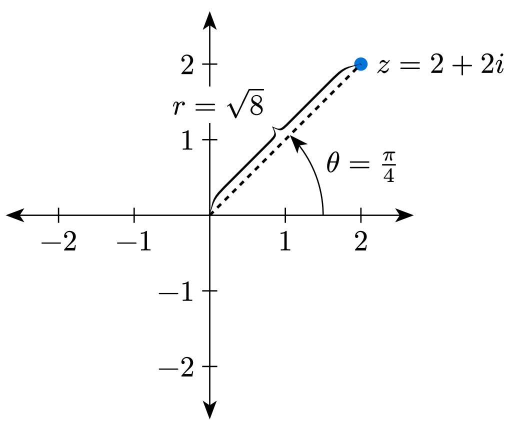
$$\frac{2i}{2+3i} = \frac{(2i)\cdot(2-3i)}{(2+3i)\cdot(2-3i)} = \frac{4i-6i^2}{13} = \frac{6}{13} + \frac{4}{13}i.$$

Euler's Formula and Polar Form for Complex Numbers

A complex number written in the form "$a + bi$" is said to be written in *rectangular form*. However, similarly to how we can represent any point in the plane in polar coordinates, we can also write a complex number in *polar form*. For any complex number z, there is a real number r and angle θ so that

$$z = r(\cos(\theta) + i\sin(\theta)).$$

We call θ the **argument** of z and write $\theta = \arg(z)$. The number r is usually positive, but is is not required to be. When it is positive, it is equal to $|z|$.



Euler's formula gives a valuable connection between a complex numbers and complex exponentials, and it's closely related to polar form.

> **Euler's Formula.** Euler's formula states that for any real number t,
>
> $$e^{it} = \cos(t) + i\sin(t).$$

Suppose $z_1 = e^{i\theta_1}$ and $z_2 = e^{i\theta_2}$ are two complex numbers. By the rules of exponents, we know

$$z_1 \cdot z_2 = e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)}.$$

More generally, if $z_1 = r_1e^{i\theta_1}$ and $z_2 = r_2e^{i\theta_2}$, then

$$z_1z_2 = r_1r_2e^{i(\theta_1+\theta_2)}.$$

This shows that when two complex numbers are multiplied, their angles are added adn their moduli are multiplied. (If $\theta = 0$ or π, then the complex number is real; test your intuition to make sure this formula does what you think for real numbers.)

Applying Complex Numbers

When solving quadratics $ax^2 + bx + c = 0$, we have a general solution in terms of the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

If $b^2 - 4ac < 0$, there is no real solution to the equation, however, there is *always* a complex solution.

Example. Solve the quadratic equation $x^2 + x + 2 = 0$.

Using the quadratic formula, we have

$$x = \frac{-6 \pm \sqrt{1^2 - 4 \cdot 1 \cdot 2}}{2 \cdot 1} = \frac{-6 \pm \sqrt{-7}}{2} = -3 \pm \frac{\sqrt{7}}{2} i.$$

In fact, all non-constant polynomials have roots in the complex numbers.

Theorem (Fundamental Theorem of Algebra)

Every non-constant polynomial with complex coefficients has roots in the complex plane and the sum of the multiplicity of those roots is equal to the degree of the polynomial.

Because of the Fundamental Theorem of Algebra, complex numbers are useful in situations involving polynomials, like eigenvalue/eigenvector problems in linear algebra.

Example. Find all eigenvalues and eigenvectors of $M = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

Computing $\det(M - \lambda I) = \lambda^2 + 1 = 0$, we see the eigenvalues of M are $\lambda = \pm i$.

To find the eigenvectors corresponding to $\lambda = i$, we compute

$$\text{null}(M - iI) = \text{null}\left(\begin{bmatrix} -i & -1 \\ -1 & -i \end{bmatrix}\right) = \text{span}\left\{\begin{bmatrix} 1 \\ -i \end{bmatrix}\right\}.$$

Thus, the eigenvectors corresponding to $\lambda = i$ are non-zero multiples of $\begin{bmatrix} 1 \\ -i \end{bmatrix}$.

Similarly, the eigenvectors corresponding to $\lambda = -i$ are non-zero multiples of $\begin{bmatrix} 1 \\ i \end{bmatrix}$.

Note, when dealing with complex matrices, it can be more difficult to spot by eye when two vectors are linearly dependent. For example, $\left\{\begin{bmatrix} -2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 & -2 \end{bmatrix}\right\}$ is a linearly *independent*, but the set $\left\{\begin{bmatrix} -i \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -i \end{bmatrix}\right\}$ is linearly *dependent*. (Verify this yourself by row reducing!)

Practice Problems

1. Explain what you need to do in two different ways.
   (a) Way 1
   (b) Way 2
2. A second question.
3. A third question.

Solutions for Appendix G

1. (a) Way 1 is good (b) Way 2 is better

2. The answer to the second question.
3.